TOPOLOGY OPTIMIZATION OF STRUCTURES UNDER STOCHASTIC EXCITATIONS

Junho Chun, Junho Song, Glaucio H. Paulino

Orlando, Florida, May 21, 2013

Department of Civil & Environmental Engineering
University of Illinois at Urbana-Champaign
Structural engineering in Natural hazards and risks

San Francisco Earthquake, 1907
http://www.documentingreality.com

Tacoma bridge, 1940
http://failuremag.com
Structural engineering in Natural hazards and risks

Courtesy of Skidmore, Owing and Merrill, LLP
Motivation

John Hancock Center
http://en.wikipedia.org/wiki/John_Hancock_Center

Ssiger International Plaza
Courtesy of Skidmore, Owing and Merrill, LLP
Motivation

- Topology optimization

Stromberg, Beghini, Baker and Paulino (2011)

Stochastic excitation

1. Discrete representation method

![Discrete representation method diagram](image)

2. Instantaneous failure probability

![Instantaneous failure probability diagram](image)

3. Stochastic topology optimization

\[
\begin{array}{l}
\min_{d} f_{obj}(\hat{\rho}(d)) \\
\text{s.t. } P(E_f) \leq P_{target} \\
\text{with } M(\hat{\rho})\ddot{u}(t,\hat{\rho}) + C(\hat{\rho})\dot{u}(t,\hat{\rho}) + K(\hat{\rho})u(t,\hat{\rho}) = f(t,\hat{\rho})
\end{array}
\]

4. Sensitivity

![Sensitivity diagram](image)

5. Numerical examples

![Numerical examples](image)
Discrete representation method

- General (random process)

\[ f(t) = \mu(t) + \sum_{i=1}^{n} v_i s_i(t) = \mu(t) + s(t)^T v \]

- \( v \): uncorrelated standard normal random variables
- \( s(t) \): deterministic basis functions based on the spectral characteristics of the process

- Example: Filtered white noise (earthquake)

\[ f(t) = \int_0^t v(\tau)s(t-\tau)d\tau \]

\[ s(t) \approx \sum_{i=1}^{n} v_i s_i(t) = \sum_{i=1}^{n} W_i \cdot h_f(t-t_i)\Delta t \]

\[ = \sum_{i=1}^{n} \sqrt{2\pi\Phi_0 / \Delta t} \cdot v_i \cdot h_f(t-t_i)\Delta t = s(t)^T v \]

Response of Linear System to Stochastic Excitation

Linear system + Gaussian

- Duhamel’s Integral
  \[ u(t) = \int_0^t f(\tau) h_s(t - \tau) d\tau \]
  \[ h_s(t) : \text{the unit-impulse response function of the system} \]

- Response
  \[ u(t) = \int_0^t \sum_{i=1}^n v_i s_i(\tau) h_s(t - \tau) d\tau = \sum_{i=1}^n v_i a_i(t) = a(t)^T v_i \]
  \[ a_i(t) = \int_0^t s_i(\tau) h_s(t - \tau) d\tau, \quad i = 1, \ldots, n \]

  (Deterministic, time-dependent)
  (Random, time-independent)

- MDOF in FEM settings

\[
\begin{bmatrix}
  u(t_1) \\
  u(t_2) \\
  \vdots \\
  u(t_{n-1}) \\
  u(t_n)
\end{bmatrix} =
\begin{bmatrix}
  u(0) \\
  u(\Delta t) \\
  u(2\Delta t) \\
  \vdots \\
  u(t_0 - \Delta t) \\
  u(t_0)
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & \cdots & 0 & v_1 \\
  0 & 0 & \cdots & v_1 & v_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & v_1 & \cdots & v_{n-2} & v_{n-1} \\
  v_1 & v_2 & \cdots & v_{n-1} & v_n
\end{bmatrix}
\begin{bmatrix}
  a_1(t_0) \\
  a_2(t_0) \\
  \vdots \\
  a_{n-1}(t_0) \\
  a_n(t_0)
\end{bmatrix}
\]

Numerical time integration

Inversely computed
Instantaneous failure probability

- ‘Instantaneous’ failure events of a linear system
  \[ E_f = \{ u(t_0) \geq u_0 \} = \{ a(t_0)^T v \geq u_0 \} \]

- Failure Probability, \( P(E_f) \)
  \[
P(u(t_0) \geq u_0) = P(u_0 - a^T(t_0)v \leq 0) = P(g(u) \leq 0)
  \]
  \[
g(u) = \frac{u_0}{\|a(t_0)\|} - \frac{a^T(t_0)v}{\|a(t_0)\|} = \beta(u_0, t_0) - \hat{u}(t_0) \cdot v
  \]
  \[
P(u(t_0) \geq u_0) = \Phi[-\beta(u_0, t_0)]
  \]
  \[
  \beta(u_0, t_0) = u_0 / \|a(t_0)\| = \hat{u}(t_0) \cdot v^*
  \]
Stochastic topology optimization formulation

\[
\min_d \quad f_{\text{obj}}(\tilde{\rho}(d)) \\
\text{s.t.} \quad P(E_f) \leq P_f^{\text{target}} \\
\text{with} \quad M(\tilde{\rho})\ddot{u}(t, \tilde{\rho}) + C(\tilde{\rho})\dot{u}(t, \tilde{\rho}) + K(\tilde{\rho})u(t, \tilde{\rho}) = f(t, \tilde{\rho})
\]

\[\tilde{\rho} : \text{Filtered density}\]
\[M : \text{Mass matrix, C: Damping matrix, K: Stiffness matrix}\]
\[f(t) : \text{Stochastic process (e.g., for earthquake loading) } f(t) = -M\dot{u}_g(t) = -Mf(t)\]
\[E_f : \text{Failure event, } P_f^{\text{target}} : \text{Target failure probability}\]
Sensitivity calculation

Computing the sensitivity of structural responses with respect to various design parameters is essential for efficient gradient-based optimization

- Probabilistic constraint on of an instantaneous failure event
  \[ P(E_f) = P(u_0 - a(t_0, \hat{p})^T v \leq 0) \leq P_f^{\text{target}} \]

- Alternatively,
  \[ \Phi[-\beta(u_0, t_0, \hat{p})] \leq \Phi(-\beta^{\text{target}}) \]

- Sensitivity of the reliability index with respect to the design variable is derived as
  \[
  \frac{\partial \beta(u_0, t_0, \hat{p})}{\partial d_c} = \frac{n}{\sum_{j=1}^{n} \frac{\partial \beta(u_0, t_0, \hat{p})}{\partial \hat{p}_j} \cdot \frac{\partial \hat{p}_j}{\partial d_c}}
  \]
  \[
  = \left[ - \frac{u_0}{(a_1(t_0, \hat{p})^2 + \cdots + a_n(t_0, \hat{p})^2)^{\frac{3}{2}}} \right] \cdot \sum_{j=1}^{n} \sum_{i=1}^{n} \left( a_i(t_0, \hat{p}) \cdot \frac{\partial a_i(t_0, \hat{p})}{\partial \hat{p}_j} \right) \cdot \frac{\partial \hat{p}_j}{\partial d_c}
  \]
  \[
  = \sum_{j=1}^{n} \sum_{i=1}^{n} \left( c_i(u_0, t_0, \hat{p}) \cdot \frac{\partial a_i(t_0, \hat{p})}{\partial \hat{p}_j} \right) \cdot \frac{\partial \hat{p}_j}{\partial d_c}
  \]

Implicitly defined term
Sensitivity calculation-cont’d

- Adjoint method by introducing adjoint system equation (Newmark method)

\[ \frac{1}{\eta(\Delta t)^2} M(\ddot{\rho}) + \frac{\gamma}{\eta \Delta t} C(\ddot{\rho}) + K(\ddot{\rho}) \left( u(t_{j+1}, \ddot{\rho}) = f(t_{j+1}, \ddot{\rho}) \right) \]

\[ + C(\ddot{\rho}) \left[ \frac{\gamma}{\eta \Delta t} u(t_j, \ddot{\rho}) + \left( \frac{\gamma}{\eta} - 1 \right) \ddot{u}(t_j, \ddot{\rho}) + \Delta t \left( \frac{\gamma}{2\eta} - 1 \right) \dddot{u}(t_j, \ddot{\rho}) \right] \]

\[ + M(\ddot{\rho}) \left[ \frac{1}{\eta(\Delta t)^2} u(t_j, \ddot{\rho}) + \frac{1}{\eta \Delta t} \ddot{u}(t_j, \ddot{\rho}) + \left( \frac{1}{2\eta} - 1 \right) \dddot{u}(t_j, \ddot{\rho}) \right] \]

- From a general recurrence relation associated with three sequential displacements (Chan et al. 1962, Zienkiewicz 1977)

\[ \left( M(\ddot{\rho}) + \gamma \Delta t C(\ddot{\rho}) + \eta (\Delta t)^2 K(\ddot{\rho}) \right) u(t_{j+1}, \ddot{\rho}) = \eta (\Delta t)^2 f(t_{j+1}, \ddot{\rho}) + (0.5 + \gamma - 2\eta)(\Delta t)^2 f(t_j, \ddot{\rho}) \]

\[ + (0.5 - \gamma + \eta)(\Delta t)^2 f(t_{j-1}, \ddot{\rho}) \]

\[ - \left[ -2M(\ddot{\rho}) + (1 - 2\gamma) \Delta t C(\ddot{\rho}) + (0.5 + \gamma - 2\eta)(\Delta t)^2 K(\ddot{\rho}) \right] u(t_j, \ddot{\rho}) \]

\[ - \left[ M(\ddot{\rho}) + (\gamma - 1) \Delta t C(\ddot{\rho}) + (0.5 - \gamma + \eta)(\Delta t)^2 K(\ddot{\rho}) \right] u(t_{j-1}, \ddot{\rho}) \]
Sensitivity calculation-cont’d

- By pre-multiplying the discretized adjoint system with the dimensional adjoint variable vector and adding to right-hand side terms of the original sensitivity equation

\[
\frac{\partial \beta(u_0, t_0, \tilde{\rho})}{\partial \tilde{\rho}_j} = \sum_{i=1}^{n} \left[ T_i \cdot z^T \frac{\partial u(t_i, \tilde{\rho})}{\partial \tilde{\rho}_j} \right] \\
+ \sum_{j=1}^{n} \lambda^T_{n-j+1} \left[ \frac{\partial A(\tilde{\rho})}{\partial \tilde{\rho}_j} \cdot u(t_j, \tilde{\rho}) - \eta (\Delta t)^2 \frac{\partial f(t_j, \tilde{\rho})}{\partial \tilde{\rho}_j} - (0.5 + \gamma - 2\eta)(\Delta t)^2 \frac{\partial f(t_{j-1}, \tilde{\rho})}{\partial \tilde{\rho}_j} \right] \\
- (0.5 - \gamma + \eta)(\Delta t)^2 \frac{\partial f(t_{j-2}, \tilde{\rho})}{\partial \tilde{\rho}_j} + \frac{\partial B(\tilde{\rho})}{\partial \tilde{\rho}_j} \cdot u(t_{j-1}, \tilde{\rho}) + \frac{\partial E(\tilde{\rho})}{\partial \tilde{\rho}_j} \cdot u(t_{j-2}, \tilde{\rho}) \\
+ \sum_{j=1}^{n} \lambda^T_{n-j+1} \left[ A(\tilde{\rho}) \cdot \frac{\partial u(t_j, \tilde{\rho})}{\partial \tilde{\rho}_j} + B(\tilde{\rho}) \cdot \frac{\partial u(t_{j-1}, \tilde{\rho})}{\partial \tilde{\rho}_j} + E(\tilde{\rho}) \cdot \frac{\partial u(t_{j-2}, \tilde{\rho})}{\partial \tilde{\rho}_j} \right]
\]

- After solving adjoint system problem

\[
\frac{\partial \beta(u_0, t_0, \tilde{\rho})}{\partial \tilde{\rho}_j} = \sum_{j=1}^{n} \lambda^T_{n-j+1} \left[ \frac{\partial A(\tilde{\rho})}{\partial \tilde{\rho}_j} \cdot u(t_j, \tilde{\rho}) - \eta (\Delta t)^2 \frac{\partial f(t_j, \tilde{\rho})}{\partial \tilde{\rho}_j} - (0.5 + \gamma - 2\eta)(\Delta t)^2 \frac{\partial f(t_{j-1}, \tilde{\rho})}{\partial \tilde{\rho}_j} \right] \\
- (0.5 - \gamma + \eta)(\Delta t)^2 \frac{\partial f(t_{j-2}, \tilde{\rho})}{\partial \tilde{\rho}_j} + \frac{\partial B(\tilde{\rho})}{\partial \tilde{\rho}_j} \cdot u(t_{j-1}, \tilde{\rho}) + \frac{\partial E(\tilde{\rho})}{\partial \tilde{\rho}_j} \cdot u(t_{j-2}, \tilde{\rho}) \\
+ \lambda^T \left[ B(\tilde{\rho}) \cdot \frac{\partial u(0, \tilde{\rho})}{\partial \tilde{\rho}_j} + E(\tilde{\rho}) \cdot \frac{\partial u(t_{j-1}, \tilde{\rho})}{\partial \tilde{\rho}_j} \right] + \lambda^T_{n-1} \left[ E(\tilde{\rho}) \cdot \frac{\partial u(0, \tilde{\rho})}{\partial \tilde{\rho}_j} \right]
\]
Flow chart for topology optimization under stochastic excitations

1. Initial Design: \( \rho = \rho_{\text{Initial}} \)
2. Generating Stochastic Excitation Model
3. Random Vibration Analysis using dynamic FEA, and compute \( P_f \)
4. Sensitivity Analysis of objective and constraint functions
5. Update Design Variables using Mathematical Programming
6. Convergence Criteria achieved?
   - No (iterate)
   - Yes
7. Result: Optimal Topology
Numerical example: Building structure


Design domain: Bilinear quadrilateral element, Q4

Frame element

Material
Concrete
Thickness
0.10 m
Time, \( t_0 \)
10 sec
Threshold values \( u_0 \)
0.02
Volf.
0.65
Column
0.4m x 0.4m
\( \zeta_f \)
0.45
\( \alpha_f \)
5\( \pi \)

Inter-story drift ratio

Filtered Gaussian process using Kanai-Tajimi filter

**Failure event - Inter-story drift ratio**

\[
E_{fi} = \left\{ \begin{array}{l}
u_{i,0} - \left( \frac{\left( a(t_0, \tilde{\rho})_{i,L} + a(t_0, \tilde{\rho})_{i,M} + a(t_0, \tilde{\rho})_{i,R} \right) v}{nL_i} \right) \leq 0 \quad \text{for } i = 2 \\
u_{i,0} - \left( \frac{\left( a(t_0, \tilde{\rho})_{i,L} + a(t_0, \tilde{\rho})_{i,M} + a(t_0, \tilde{\rho})_{i,R} \right) v}{nL_i} - \frac{\left( a(t_0, \tilde{\rho})_{(i-1),L} + a(t_0, \tilde{\rho})_{(i-1),M} + a(t_0, \tilde{\rho})_{(i-1),R} \right) v}{nL_i} \right) \leq 0 \quad \text{for } i = 3, 4, \ldots, n \end{array} \right.
\]

**Filtered Gaussian process using KT filter**

- Kanai-Tajimi filter model

\[
h_f(t) = \exp(-\zeta_f \omega_j t) \left[ \frac{(2\zeta_f^2 - 1)\omega_j}{\sqrt{1 - \zeta_f^2}} \sin(\omega_j \sqrt{1 - \zeta_f^2} \cdot t) - 2\zeta_f \omega_j \cos(\omega_j \sqrt{1 - \zeta_f^2} \cdot t) \right]
\]

\[
\Phi(\omega) = \frac{1 + 4\zeta_f^2(\omega / \omega_g)^2}{[1 - (\omega / \omega_g)^2]^2 + (2\zeta_f \omega / \omega_g)^2} \Phi_0
\]

\[
f(t) = \sum_{i=1}^{n} \sqrt{2\pi \Phi_0 / \Delta t} \cdot v_i \cdot h_f(t - t_i) \Delta t
\]

Parametric study on impact of ground motion characteristics

- $\omega_f = 5.0\pi$
- $\zeta_g = 0.3$
- $\zeta_g = 0.4$
- $\zeta_g = 0.5$

- $\omega_f = 4.5\pi$
- $\omega_f = 5.0\pi$
- $\omega_f = 5.5\pi$

### Numerical examples

<table>
<thead>
<tr>
<th>Case</th>
<th>Design Parameters</th>
<th>Volume</th>
<th>Reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$(\Phi_0=3, \beta_{\text{target}}=1.5)$</td>
<td>$2.81 \text{ m}^3$ (18.7%)</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>$(\Phi_0=3, \beta_{\text{target}}=1.5)$</td>
<td>$2.42 \text{ m}^3$ (16.1%)</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>$(\Phi_0=3, \beta_{\text{target}}=1.5)$</td>
<td>$2.27 \text{ m}^3$ (15.1%)</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>$(\Phi_0=3, \beta_{\text{target}}=1.5)$</td>
<td>$2.08 \text{ m}^3$ (13.9%)</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>$(\Phi_0=3, \beta_{\text{target}}=1.5)$</td>
<td>$2.42 \text{ m}^3$ (16.1%)</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>$(\Phi_0=3, \beta_{\text{target}}=1.5)$</td>
<td>$2.82 \text{ m}^3$ (18.8%)</td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

- Introduction of a new approach incorporating random vibration theories into topology optimization using a discrete representation method for stochastic processes

- Computation of the failure probability regarding stochastic responses by a closed-form solution

- Implementation of the Adjoint method to evaluate the sensitivity of dynamic responses modeled by the discrete representation method

- Application of the proposed stochastic topology optimization method to achieve optimal solutions satisfying probabilistic constraints on stochastic responses of structural systems
Acknowledgements

Prof. Junho Song
Derya Deniz
Hyun-woo Lim
Nolan Kurtz
Roselyn Jihyen Kim
Reece Otsuka

Prof. Glauco H. Paulino
Arun Gain
Tomas Zegard
Sofie Leon
Daniel Spring
Evgueni Filipov
Heng Chi
Maryam Eidini

Prof. Adeildo Soares Ramos Jr.
Prof. Ivan Menezes
Tam Nguyen
Lauren Beghini
Cameron Talischi

- Karol Fellowship
- National Science Foundation

Question and Comments?
Performance of sensitivity methods

- Input stochastic process

\[ f(t) = s(t)^T v \quad s_i(t) = \exp[-2.4\pi(t - t_i)] \sin[3.2\pi(t - t_i)]H(t - t_i) / \|s(t)\| \]

- Failure event

\[ E_f = u_0 - \left( \frac{(a(t_0, \tilde{\rho})^T_{\text{Left}} + a(t_0, \tilde{\rho})^T_{\text{Right}})v}{2} \right) \leq 0 \]

- Sensitivity

\[ \frac{\partial \beta(u_0, t_0, \tilde{\rho})}{\partial d_e} \]

---

<table>
<thead>
<tr>
<th>E</th>
<th>21,000Mpa</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>2,400kg/m³</td>
</tr>
<tr>
<td>Thickness</td>
<td>0.10 m</td>
</tr>
<tr>
<td>Column</td>
<td>0.35m x 0.35m</td>
</tr>
<tr>
<td>Time, ( t_0 )</td>
<td>7 sec</td>
</tr>
<tr>
<td>Threshold values ( u_0 )</td>
<td>0.02</td>
</tr>
<tr>
<td>Initial.Volf</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Performance of sensitivity methods-cont’d

Direct differentiation method  Adjoint method

Finite difference method

Normalized computational time

FDM  DDM  AJM

a 200 finite elements, b 800 finite elements, and c 1800 finite elements

a  b  c
Reliability-Based Design Optimization

- Deterministic design optimization (DO)

$$\min_{d} f_{obj}(d)$$
$$s.t\quad g_i(d) > 0, \quad i = 1, \ldots, n_c$$
$$d_{lower} \leq d \leq d_{upper}$$

- Reliability-based design optimization (RBDO)

$$\min_{d, \mu_X} f_{obj}(d, \mu_X)$$
$$s.t\quad P(E_{sys}) = P[\bigcup g_i(d, \mu_X) \leq 0] \leq P_{sys}^{target}, \quad i = 1, \ldots, n_c$$
$$d_{lower} \leq d \leq d_{upper}, \quad \mu_{lower} \leq \mu_X \leq \mu_{upper}$$
System Reliability-Based Design Optimization

**Component Reliability Based Design Optimization (CRBDO)**

\[
\begin{align*}
\min_{d, \mu_X} & \quad f_{\text{obj}}(d, \mu_X) \\
\text{s.t.} & \quad P\left[g_i(d, \mu_X) \leq 0\right] \leq P_{f}^{\text{target}}, \quad i = 1, \ldots, n_c \\
& \quad d_{\text{lower}} \leq d \leq d_{\text{upper}}, \quad \mu_{\text{lower}} \leq \mu_X \leq \mu_{\text{upper}}
\end{align*}
\]

**System Reliability Based Design Optimization (SRBDO)**

\[
\begin{align*}
\min_{d, \mu_X} & \quad f_{\text{obj}}(d, \mu_X) \\
\text{s.t.} & \quad P(E_{\text{sys}}) = P\left[\bigcup g_i(d, \mu_X) \leq 0\right] \leq P_{\text{sys}}^{\text{target}}, \quad i = 1, \ldots, n_c \\
& \quad d_{\text{lower}} \leq d \leq d_{\text{upper}}, \quad \mu_{\text{lower}} \leq \mu_X \leq \mu_{\text{upper}}
\end{align*}
\]

MSR method  
(Song and Kang 2009, Kang et al. 2012)

---


**Optimal topologies**

- Varying target failure probabilities

<table>
<thead>
<tr>
<th>Level</th>
<th>EL (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>05</td>
<td>20</td>
</tr>
<tr>
<td>04</td>
<td>15</td>
</tr>
<tr>
<td>03</td>
<td>10</td>
</tr>
<tr>
<td>02</td>
<td>5</td>
</tr>
<tr>
<td>Ground</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case</th>
<th>Target Failure Probability</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\beta_{\text{target}} = 1.4 (P_f = 8.1%)$</td>
<td>$4.95 \text{ m}^3 (24.7%)$</td>
</tr>
<tr>
<td>B</td>
<td>$\beta_{\text{target}} = 1.7 (P_f = 4.5%)$</td>
<td>$6.15 \text{ m}^3 (30.8%)$</td>
</tr>
<tr>
<td>C</td>
<td>$\beta_{\text{target}} = 2.0 (P_f = 2.3%)$</td>
<td>$7.19 \text{ m}^3 (35.9%)$</td>
</tr>
<tr>
<td>D</td>
<td>$\beta_{\text{target}} = 2.3 (P_f = 1.1%)$</td>
<td>$8.34 \text{ m}^3 (41.7%)$</td>
</tr>
</tbody>
</table>
Convergence and Dynamic response

- **Convergence history**
  - Graph showing convergence history with plots for $\beta$, volume, and $P_f$.

- **Dynamic response comparison**
  - Graph showing dynamic response comparison with plots for $f(t)$ and $\Delta L_i$ for $i = 1, 2, 3, 4$.

---

**Introduction**

- Discrete representation method
- Instantaneous failure probability
- Stochastic topology optimization
- Sensitivity

**Optimization procedure**

**Numerical examples**