On Optimization of Shape and Topology
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Introduction:

- The goal of optimal shape design is to find the most efficient shape of a physical system.
- The response is captured by the solution \( u_\omega \) to a boundary value problem that in turn depends on the given shape \( \omega \)

\[
\inf_{\omega \in \Omega} J(\omega, u_\omega) \quad \text{where} \quad B(u_\omega, v; \omega) = \ell(v), \quad \forall v \in V
\]

\[
\begin{align*}
B(u, v; \omega) &= \int_\Omega \nabla u : [\chi_\omega C^t + (1 - \chi_\omega) C] : \nabla v \, dx, \\
\ell(v) &= \int_{\Gamma_N} t \cdot v \, ds
\end{align*}
\]

Restriction setting:

- If \( \chi_n, \chi \in L^\infty(\Omega; [0, 1]) \) such that \( \chi_n \to \chi \) in \( L^1(\Omega) \), then, up to a subsequence, the associated state solutions also converge, i.e., \( u_{\chi_n} \to u_\chi \) in \( H^1(\Omega; \mathbb{R}^d) \)
- It follows that compactness in \( L^1(\Omega) \) topology is a sufficient condition for existence of solutions.
- A well-known example is the space of shapes with bounded perimeter:

\[
\mathcal{A} = \{ \chi \in BV(\Omega; [0, 1]): \int_\Omega |\nabla \chi| \, dx \leq \mathcal{T} \}
\]

Continuous parametrization:

- \( \mathcal{A} \subseteq L^\infty(\Omega; [0, 1]) \)
- Regularization
- Regularization
- Same solution?
- Depends!
- \( \mathcal{A} \subseteq L^\infty(\Omega; [0, 1]) \)
- (Compact)
- (Compact)

Optimization problem:

- Composite objective: \( \min_{\rho \in \mathcal{A}} F(\rho) := J(\rho) + R(\rho) \)
- Performance functional: \( J(\rho) = \int_{\Gamma_N} t \cdot u_\rho \, ds + \lambda \int_\Omega \rho \, dx \)
- Regularizer: \( R(\rho) = \beta \frac{1}{2} \int_\Omega |\nabla \rho|^2 \, dx = \frac{1}{2} (\rho, R \rho), \quad R = -\beta \Delta \)
- Admissible densities: \( \mathcal{A} = \{ \rho \in H^1(\Omega): 0 \leq \rho \leq 1 \} \)
- State equation: \( \int_\Omega \nabla u_\rho : C_p : \nabla v \, dx = \int_{\Gamma_N} t \cdot v \, ds, \quad \forall v \in V \)

Forward-backward splitting algorithm:

- We consider an optimization algorithm of the form:

\[
\rho_{n+1} = \arg\min_{\rho \in \mathcal{A}} \frac{1}{2\tau_n} \| \rho - [\rho_n - \tau_n J'(\rho_n)] \|^2 + R(\rho)
\]

The intuition is that the next iterate \( \rho_{n+1} \) is close to the gradient descent update on \( J \), i.e., \( \rho_{n+1} - \tau_n J'(\rho_n) \), while minimizing the regularizer \( R(\rho) \)
- Given constants \( \tau_0 > 0 \) and \( 0 < \sigma < 1 \), the step size parameter is set to be

\[
\tau_n = \sigma^k \tau_0
\]

where \( k_n \) is the smallest non-negative integer such that \( \tau_n \) satisfies

\[
F(\rho_n) - F(\rho_{n+1}) \geq \frac{1}{2\tau_n} \| \rho_n - \rho_{n+1} \|^2
\]

Improving convergence:

- We consider the following generalization:

\[
\rho_{n+1} = \arg\min_{\rho \in \mathcal{A}} J(\rho_n) + J'(\rho_n) \rho + \frac{1}{2\tau_n} \rho^2 \equiv \rho - \tau_n H_n(\rho - \rho_n) + R(\rho)
\]

where \( H_n \) is a bounded linear positive-definite operator.
- The reciprocal approximation of compliance is its Taylor expansion in the intermediate field \( \rho^{-1} \)

\[
J_{rec}(\rho; \rho_n) = J(\rho_n) + J'(\rho_n) \rho - \frac{1}{2} \left( \rho - \rho_n, \frac{2E(\rho_n)}{\rho} (\rho - \rho_n) \right)
\]

where \( E(\rho) \equiv \rho^{-1}[\nabla u_\rho : (C^t - C) : \nabla u_\rho] \) is the gradient of compliance.
- We embed the same type of approximation into our quadratic model by setting

\[
H_n = J_{rec}(\rho; \rho_n) = \frac{2E(\rho_n)}{\rho} I
\]

Performance of the algorithm:

<table>
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<th>Algorithm</th>
<th>( H_n )</th>
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<th># BT</th>
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