ON RESTRICTION METHODS FOR TWO-PHASE OPTIMAL SHAPE PROBLEMS

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Motivation

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The main motivation of this talk is to explore the existence issue, within the restriction framework, for the implicit function description of the problem.

For example, we show that a consequence of the ill-posedness is that smearing of Heaviside function transforms the topology problem into the variable thickness problem.
The two-phase optimal shape problem is defined as:

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\inf_{\chi \in \mathcal{A}} J(\chi, u_{\chi}) \quad \text{where } u_{\chi} \in \mathcal{V} \text{ solves } B(u, v; \chi) = \ell(v), \quad \forall v \in \mathcal{V}
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\( \mathcal{A} \subseteq L^\infty(\Omega; \{0, 1\}) \) is the given space of admissible designs,

\[ \mathcal{B}(u, v; \chi) = \int_{\Omega} \varepsilon(u) : [\chi C^+ + (1 - \chi) C^-] : \varepsilon(v) \, dx, \quad \ell(v) = \int_{\Gamma_N} t \cdot v \, ds \]

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The two-phase optimal shape problem is defined as:

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- The objective function for the minimum compliance is given by

$$J(\chi, u_\chi) = \ell(u_\chi) + \lambda \int_\Omega \chi \, dx$$

where $\lambda$ is the volume penalty parameter
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$$J(\chi, u_\chi) = \ell(u_\chi) + \lambda \frac{1}{2} \int_\Omega \chi dx, \quad \Gamma_D = \emptyset, \quad t = (e_d \otimes n) \cdot t_0 e_d$$

Let $\varphi_n(x) = \alpha \sin(nx_1)$. Then $\chi_n = H(\varphi_n)$ is a minimizing sequence that does not converge to an element of $A$.
Ill-posedness

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- The optimal design for this problem is a rank-1 laminate, whose stiffness is precisely the \( H \)–limit of \( \chi_n C^+ + (1 - \chi_n) C^- \)
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**PROPOSITION:** Let $\chi_n, \hat{\chi} \in L^\infty(\Omega; [0, 1])$ be such that $\chi_n \to \hat{\chi}$ in $L^1(\Omega)$. Then, up to a subsequence, the associated state solutions also converge, i.e., $u_{\chi_n} \to u_{\hat{\chi}}$ in $H^1(\Omega)$. 
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It follows that compactness in $L^1(\Omega)$ topology is a sufficient condition for existence of solutions:

- Given a minimizing sequence $\chi_n$, one can extract a convergent subsequence such that $\chi_n \to \hat{\chi}$ and $J(\chi_n, u_{\chi_n}) \to J(\hat{\chi}, u_{\hat{\chi}})$.
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A well-known example is the space of designs with bounded perimeter:

$$\mathcal{A} = \{ \chi \in BV(\Omega \{0, 1\}) : \int_\Omega |\nabla \chi| \, dx \leq P \}$$
Another choice (Liu et al. 2003) is to set $A = H(F)$ where the implicit functions $\varphi \in F \subseteq W^{1+\theta,2}$ satisfy:

\begin{align*}
(R1) & : \|\varphi\|_{W^{1+\theta,2}(\Omega)} \leq M \\
(R2) & : |\varphi(x)| + |\nabla \varphi(x)| \geq \nu \quad \text{a.e. } x \in \Omega
\end{align*}

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(R2) ensures that the phase boundary

$$\{x \in \Omega : \varphi(x) = 0\}$$

which is where the Heaviside is discontinuous, has zero measure:

- Without it, $\varphi_n(x) = (\alpha/n^{2+\theta}) \sin(nx_1)$ gives a minimizing sequence that satisfies (R1) but does not converge
If no restrictions are placed on \( \varphi \), the usual approximation of the Heaviside by

\[
H_w(\varphi)(x) = \begin{cases} 
0, & \varphi(x) < -w \\
h_w(\varphi(x)), & |\varphi(x)| \leq w \\
1, & \varphi(x) > w
\end{cases}
\]

transforms the problem into the \textit{variable thickness problem} regardless of \( w \):
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- For any $\rho \in L^\infty(\Omega; [0, 1])$, there exists $\varphi \in L^\infty(\Omega; [-\alpha, \alpha])$ such that $\rho = H_w(\varphi)$. Conversely, $H_w(\varphi)$ represents a thickness function.
- Note also that the conditions of optimality are the same:

$$H'_w(\varphi) [\lambda - E(u)] = 0 \quad \text{when} \quad -w < \varphi < w$$

where $E(u) = \epsilon(u) : (C^+ - C^-) : \epsilon(u)$.
Approximation of the Heaviside

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Therefore the optimal solution with such approximation will contain large "grey" regions filled by the intermediate phases.
□ (R1) can be imposed via convolution with a smooth filter, i.e., by defining
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\mathcal{F} = \{K \ast \eta : \eta \in L^\infty(\Omega; [-\alpha, \alpha])\}
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Smoothness, transversality

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- To impose transversality, we can augment the objective function
  \[ J_\beta(\chi, u_\chi) = J(\chi, u_\chi) + \beta \int_\Omega \chi (1 - \chi) \, dx \]
  OR change the state equation to penalize the intermediate stiffnesses:
  \[ B_p(u, v; \chi) = \int_\Omega \epsilon(u) : [\chi^p C^+ + (1 - \chi^p) C^-] : \epsilon(v) \, dx \]
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  In both cases, separation of phases and thus transversality is achieved in the optimal regime.

- The condition of optimality for \{ -w < \varphi < w \} respectively are:
  \[ H'_w(\varphi) \{ \lambda + \beta [1 - 2H_\omega(\varphi)] - E(u) \} = 0 \]
  \[ H'_w(\varphi) \{ \lambda - p [H_w(\varphi)]^{p-1} E(u) \} = 0 \]
The continuum parameters (i.e., those independent of the mesh size) are:

- $\alpha$: bound for implicit function field
- $R$: radius of filtering kernel $K$
- $w$: width of the approximate Heaviside
- $p$: parameter for penalization of intermediate stiffnesses

It is not easy to establish an explicit relationship between $\nu$ with above parameters in general.

However the compliance problem, the transversality constant $\nu$ is directly related to $\alpha/R$ (which is why we set $w$ to be fixed fraction of $\alpha/R$).
Some numerical results

Initial guess:

Values of parameters used: \( \alpha = 1, \ w = 0.0375\alpha/R, \ p = 4 \)
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\[ H_w(\varphi) \]

\[ \eta \]

\[ \varphi = K_R \ast \eta \]

\[ R = 0.075 \]
Some numerical results

$H_w(\varphi)$

$\eta$

$\varphi = K_R \ast \eta$

$R = 0.075$

$R = 0.100$
Some numerical results

\[ H_w(\varphi) \quad \eta \quad \varphi = K_R * \eta \]

\( R = 0.075 \)

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\( R = 0.150 \)
Some numerical results

\[ H_w(\varphi) \]

\[ \eta \]

\[ \varphi = K_R * \eta \]

\[ R = 0.075 \]

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\[ R = 0.200 \]
Some numerical results

Initial guess

Final solution
Concluding remarks

□ The nature of the continuum optimal shape problem has implications for the numerical formulations and algorithms

□ In addition to smoothness, a uniform “transversality” condition must be imposed on the implicit function field

□ Within the restriction framework, the Ersatz material model (filling the voids with compliant material $C^-$) can be justified
This fact can be illustrated numerically:

With transversality condition (R2) imposed, however, we can prove that as \( w \to 0 \), the optimal solution \( \chi_w^* = H_w(\varphi^*) \) converge to solution of problem with \( A = H(F) \).