A unified library of nonlinear solution schemes

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7/27/2011
Motivation

- Nonlinear problems are prevalent in structural, fluid, continuum, etc. mechanics.

- Nonlinear wave interaction simulated with libMesh. [libMesh wiki](https://libmesh.github.io).


Motivation

- Many algorithms have been developed to solve such problems, but no single algorithm is capable of solving any and all nonlinear problems.

\[ f(u) = -3\text{sign}(u)|u|^3 + 4u + 1 \]

- A library of nonlinear solution schemes, defined by unique constraint equations, is unified into a single space.

7/27/2011  
USNCCM-11 | A unified library of nonlinear solution schemes | Sofie E. Leon
Presentation Outline

• Nonlinearity overview

• N+1 dimensional space formulation

• Solution schemes

• Numerical results

• Evaluation and conclusions
Nonlinearity overview

- **Geometric**
  - Large strains and/or rotations
  - Linear or nonlinear constitutive relation
  - Loads have an effect on deformed configuration or configuration can have an effect on the load

- **Material**
  - i.e. inelasticity, strain-softening

- **Critical points**
  - Structure loses stability or bifurcation occurs
External loads and displacements
\[ u_j = u_{prev}^j + \Delta u_{j-1}^j + \delta u_j \]
\[ p_j = p_{prev}^j + \Delta p_{j-1}^j + \delta p_j \]

Residual
\[ r_j = p_{prev}^j + \Delta p_j^j - q(u_{prev}^j + \Delta u_j) \]

Governing equation
\[ K_{j-1} \delta u_j = p_j - q_{j-1} \]

Load parameter and reference load vector
\[ p_j = p_{prev}^j + \Delta p_{j-1}^j + \delta \lambda_j \bar{p} \]

N+1 space
\[ a_j \cdot \delta u_j + b_j \delta \lambda_j = c_j \]
\[ \begin{bmatrix} K_{j-1} & -\bar{p} \\ (a_j)^T & b_j \end{bmatrix} \begin{bmatrix} \delta u_j \\ \delta \lambda_j \end{bmatrix} = \begin{bmatrix} r_{j-1}^p \\ c_j \end{bmatrix} \]

Decomposition
\[ \delta u = \delta u_I + \delta \lambda \delta u_{II} \]
\[ K_{j-1} \delta u_{II}^j = \bar{p} \]

Load parameter
\[ \delta \lambda_j = \frac{c_j - a_j \cdot \delta u_{III}^j}{a_j \cdot \delta u_{III}^j + b_j} \]

Unified scheme – NLS++

- Incremental-iterative scheme is the same for any algorithm implemented in the N+1 space

- Only the computation of the load factor changes for each algorithm

\[ \lambda' \leftarrow \lambda' + \delta \lambda'_{ij} \]
\[ u' \leftarrow u' + (\delta \lambda'_{ij} \delta u'_{ij} + \delta u'_{ii}) \]
\[ p' \leftarrow \lambda' \tilde{p} \]
\[ r'_{ij} \leftarrow p' - q(u') \]
\[ \frac{\|r'_{ij}\|}{\|\Delta \lambda' p\|} < \text{TOL} \]
\[ i \leftarrow i + 1 \]
\[ j \leftarrow j + 1 \]
Nonlinear solution schemes

- $\delta \lambda$ is computed for each nonlinear solution scheme based on its unique constraint
  - Load control method (LCM)
  - Displacement control method (DCM)
  - Arc-Length control method (ALCM)
  - Work control method (WCM)
  - Generalized displacement control method (GDCM)
  - Orthogonal residual procedure (ORP)
Common nonlinear solution schemes

Load control method

Arc-length control method

Constraint Surface

$\lambda \vec{p}$

$\Delta \lambda \vec{p}$

$\Delta \lambda p$

$\Delta u$

$u$

Constraint Surface

$\lambda \vec{p}$

$\Delta \lambda_1 \vec{p}$

$\Delta \lambda_2 \vec{p}$

$\Delta \lambda_3 \vec{p}$

$\Delta s^i$

$\Delta u_1$

$\Delta u_2$

$\Delta u_3$
Generalized displacement control method

• Load parameter is based on a physical quantity called the Generalized Stiffness Parameter (GSP)

\[
GSP = \frac{\delta u^1_{1_1} \cdot \delta u^1_{1_1}}{\delta u^{i-1}_{1_1} \cdot \delta u^i_{1_1}}
\]

• The sign of the GSP changes only at load limit points

Yang, Y.B. and Sheih, M.S. AIAA Journal 1990.
Two-dimensional function

- Internal force given by
  \[ q(u) = \begin{pmatrix} 10u_1 + 0.4u_2^3 - 5u_2^2 \\ 0.4u_1^3 - 3u_1^2 + 10u_2 \end{pmatrix} \]

- Tangent matrix
  \[ K(u) = \begin{bmatrix} 10 & 1.2u_2^2 - 10u_2 \\ 1.2u_1^2 - 6u_1 & 10 \end{bmatrix} \]

LCM = load control method
DCM = displacement control method
ALCM = arc-length control method
WCM = work control method
GDCM = generalized displacement control method
ORP = orthogonal residual procedure
var = variable
u1, u2 = control degrees of freedom

Ilinca Stanciulescu. Personal communications. 2009.
Two-dimensional function and the variable DCM

- Systematically select the best control parameter

<table>
<thead>
<tr>
<th>Step</th>
<th>Control</th>
<th>Snap back</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-73</td>
<td>$u_1$</td>
<td>Step 58 in $u_2$, Label 1</td>
</tr>
<tr>
<td>74-216</td>
<td>$u_2$</td>
<td>Step 158 in $u_1$, Label 2</td>
</tr>
<tr>
<td>217-445</td>
<td>$u_1$</td>
<td>Step 266 in $u_2$, Label 3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Step 398 in $u_2$, Label 4</td>
</tr>
</tbody>
</table>

Twelve bar truss

Krenk S and Hededal O. *CMAME* 1995.

NRM = newton raphson method, DCM = displacement control method, ALCM = arc-length control method, WCM = work control method, GDCM = generalized displacement control method, ORP = orthogonal residual procedure
Lee frame

\[ E, A \]

\[ I, G \]

\[ P \]

\[ x \]

\[ y \]

\[ u_1 \]

\[ u_2 \]

\[ 120 \]

\[ 120 \]

Schweizerhof KH and Wriggers P. *CMAME* 1986.
Parente E and Vaz LE. *IJNME* 2001

LCM = load control method
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Evaluation of solution schemes

• In general the most robust and powerful solution schemes in the context of the N+1 space are
  – Variable displacement control method
  – Arc-length control method
  – Generalized displacement control method

• Standard displacement control method and work control method have predictable difficulties at displacement limit points

• Orthogonal residual procedure is sensitive to input parameters and behavior is sometimes unpredictable near limit points
Conclusions and future work

• NLS++ is a powerful computational framework that provides seamless implementation of solution schemes of interest
  – Gives user flexibility to select the most appropriate scheme for a particular problem
  – Portable, effective and extendable

• Plan to integrate NLS++ with the new finite element code based on topological data representation, called TopFEM

• Details and code available in M.S. thesis
  
  www.ideals.illinois.edu
  
  www.sofieleon.com