Previous Class

Isoparametric elements (cont’d)
- Mapping of elements
- 4-node bilinear quadrilateral
- Jacobian matrix
- Understanding of $[J]$ and $J$
- Examples
- Stiffness matrix
- Q8, T3, T6

This Class

Numerical integration: Gauss quadrature
- Introduction to numerical integration and Gauss quadrature
- Gauss quadrature in one dimension (1D)
- Derivation of Gauss points and weights
  (one point, two-point, and $n$ point quadrature)
- FEM example
- Gauss quadrature in 2D
- Exactness of Gauss quadrature in 2D
- Full, reduced and recommended integration in 2D
- Minimum integration order
- Gauss quadrature in 3D
Introduction

Numerical integration:

• Evaluating the integrand at specific points
• Multiplying each result by an appropriate weighting factor
• Summing up the results

\[ \int_{x_1}^{x_2} f(x) \, dx \approx W_1 f(x_1) + W_2 f(x_2) + \ldots + W_n f(x_n) \]

The choice of sampling points and weights defines different quadrature schemes.

Gauss quadrature (a numerical integration scheme)

• Formulated to compute exact integration for polynomials

• Use fewer sampling points compared to other integration schemes

• Is the most used numerical integration scheme to obtain element stiffness matrix in FEM
One dimension Gauss quadrature

Transform of integration limit: from $x_1$ and $x_2$ to -1 and 1 so that the formulas will be generalized

$$x = \frac{1}{2}(1 - \xi)x_1 + \frac{1}{2}(1 + \xi)x_2$$

$$\int_{x_1}^{x_2} f dx \quad \Rightarrow \quad \int_{-1}^{1} \phi d\xi$$

$$\phi = f\left(x(\xi)\right) \frac{dx}{d\xi}$$

Transformation Jacobian \( J = \frac{dx}{d\xi} = \frac{x_2 - x_1}{2} \)

From now on, we need to work only with -1 to 1 limits

One dimension Gauss quadrature

Assume the integrand is a polynomial

$$\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + \ldots + a_n\xi^n$$

Exact integration for each terms:

$$\int_{-1}^{1} d\xi = 2$$

$$\int_{-1}^{1} \xi^2 d\xi = 2 / 3$$

$$\int_{-1}^{1} \xi^4 d\xi = 2 / 5$$

$$\int_{-1}^{1} \xi d\xi = 0$$

$$\int_{-1}^{1} \xi^3 d\xi = 0$$

...  

Remarks: integration of odd order terms are always zero
Deriving sampling points and weights

Based on the condition that all terms need to be integrated exactly

Example First order polynomial

\[ f(\xi) = a_0 + a_1 \xi \]

\[
\int_{-1}^{1} a_0 d\xi = 2a_0 = \sum_{i=1}^{n} W_i a_0 \xi_i^0
\]

\[
\int_{-1}^{1} a_1 \xi d\xi = 0 = \sum_{i=1}^{n} W_i \xi_i
\]

There are two equations;
- We need at least two unknowns for the solution to exist
- At least 1 Gauss point needed (n=1)

One Gauss point rule (n=1)

Use 1 Gauss point: two unknown \( W_i \) and \( \xi_i \)

\[
\int_{-1}^{1} a_0 d\xi = 2a_0 = W_i a_0 \Rightarrow W_i = 2
\]

\[
\int_{-1}^{1} a_1 \xi d\xi = 0 = W_i \xi_i \Rightarrow \xi_i = 0
\]

Answer: one Gauss point

\[ \xi_i = 0 \quad W_i = 2 \]

Geometric interpretation
Two Gauss point rule (n=2)

Example: Third order polynomial

\[ \phi(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 \]

\[
\int_{-1}^{1} a_0 d\xi = 2a_0 = \sum_{i=1}^{n} W_i a_0
\]

\[
\int_{-1}^{1} a_1 \xi d\xi = 0 = \sum_{i=1}^{n} W_i a_1 \xi_i
\]

\[
\int_{-1}^{1} a_2 \xi^2 d\xi = 2/3a_2 = \sum_{i=1}^{n} W_i a_2 \xi_i^2
\]

\[
\int_{-1}^{1} a_3 \xi^3 d\xi = 0 = \sum_{i=1}^{n} W_i a_3 \xi_i^3
\]

There are 4 equations;
Need at least 4 unknowns for the solution to exist
So, at least 2 Gauss points needed

Unknown: \( W_1, \xi_1, W_2, \xi_2 \)

\[
W_1 + W_2 = 2
\]

\[
W_1 \xi_1 + W_2 \xi_2 = 0
\]

\[
W_1 \xi_1^2 + W_2 \xi_2^2 = 2/3
\]

\[
W_1 \xi_1^3 + W_2 \xi_2^3 = 0
\]

Solving the equations, we have:

\[
W_1 = 1
\]

\[
W_2 = 1
\]

\[
\xi_1 = \sqrt{1/3}
\]

\[
\xi_2 = -\sqrt{1/3}
\]
Three Gauss point rule (n=3)

Fifth order polynomial

\[ \phi(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + a_4 \xi^4 + a_5 \xi^5 \]

\[ \int_{-1}^{1} a_0 d\xi = 2a_0 = \sum_{i=1}^{n} W_i a_0 \]

\[ \int_{-1}^{1} a_1 \xi d\xi = \frac{2}{3}a_2 = \sum_{i=1}^{n} W_i a_2 \xi_i^2 \]

\[ \int_{-1}^{1} a_3 \xi^3 d\xi = \frac{2}{5}a_4 = \sum_{i=1}^{n} W_i a_4 \xi_i^4 \]

There are 6 equations;
Need at least 6 unknowns for the solution to exist
So, at least 3 Gauss points needed

Unknown:

\[ W_1, \xi_1, W_2, \xi_2, W_3, \xi_3 \]

\[ W_1 + W_2 + W_3 = 2 \]
\[ W_{1\xi_1} + W_{2\xi_2} + W_{3\xi_3} = 0 \]
\[ W_{1\xi_1^2} + W_{2\xi_2^2} + W_{3\xi_3^2} = 2 / 3 \]

Solving the equations, we have:

\[ W_1 = 5 / 9 \quad \xi_1 = -\sqrt{3 / 5} \]
\[ W_2 = 8 / 9 \quad \xi_2 = 0 \]
\[ W_3 = 5 / 9 \quad \xi_3 = \sqrt{3 / 5} \]
Remarks on the derivation

1) Note that the integration for odd order terms are always zero, e.g.

\[ W_1ξ_1 + W_2ξ_2 = 0 \]
\[ W_1ξ_3 + W_2ξ_2 = 0 \]

This condition is equivalent to the symmetry of the Gauss points and weights

2) In general, polynomials of order 2n-1 is integrated exactly by n Gauss point rule

Bonus HW: derive the four point rule
Bonus HW: derive the five point rule

FEM example

Beam element \( (X = ξ, I) \)

\[
[K] = \int_{length} [B]^T [D][B] dx
\]

\[ [B] = \frac{1}{2}[N_{1,ξ} \quad N_{2,ξ} \quad N_{3,ξ} \quad N_{4,ξ}] \]

Polynomial order 1

\[
[K] = \int_{-1}^{1} [B]^T [EI][B] l dξ
\]

Polynomial order 2

( use two point Gauss quadrature )

Recall: cubic shape function

\[
[K] = (1)[B]^T [EI][B] \bigg|_{-\sqrt{1/3}}^{\sqrt{1/3}} + (1)[B]^T [EI][B] \bigg|_{-\sqrt{1/3}}^{\sqrt{1/3}}
\]
**Gauss quadrature in two dimensions**

\[ I = \int \int_{-1,1} \phi(\xi, \eta) d\xi d\eta \approx \int \int_{-1,1} \sum W_i \phi(\xi_i, \eta) d\eta \]

\[ = \sum W_i \left[ \sum W_j \phi(\xi_j, \eta) \right] = \sum W_i \sum W_j \phi(\xi_i, \eta_j) \]

\[
\begin{bmatrix}
\phi_{11} & \phi_{12} & \phi_{13} \\
\phi_{21} & \phi_{22} & \phi_{23} \\
\phi_{31} & \phi_{32} & \phi_{33}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sum W_i \phi(\xi_i, \eta) \\
\sum W_i \phi(\xi_i, \eta)
\end{bmatrix}
\]

- **2 point rule**
  - m x n rule possible but not recommended

---

**Exactness of Gauss quadrature in 2D**

\[ I = \int \int_{-1,1} \xi^l \eta^m d\xi d\eta \]

**Constant** (l = m = 0)

**Linear** (l + m = 1)

**Quadratic** (l + m = 2)

**Cubic** (l + m = 3)

**Quartic** (l + m = 4)

**Gauss rule for exact Integration**

- **One point**
  - 2x2

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[Diagram showing Gauss quadrature rules and their exactness in 2D integration]
Full, reduced and recommended integration in 2D

Table 6.8-1 Cook et al. 2nd Ed. pp226

<table>
<thead>
<tr>
<th>Element</th>
<th>Full</th>
<th>Reduced</th>
<th>Recommended</th>
</tr>
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<tbody>
<tr>
<td>Q4</td>
<td>2x2</td>
<td>1</td>
<td>2x2</td>
</tr>
<tr>
<td>Q8</td>
<td>3x3</td>
<td>2x2</td>
<td>2x2</td>
</tr>
<tr>
<td>Q9</td>
<td>3x3</td>
<td>2x2</td>
<td>2x2 (</td>
</tr>
<tr>
<td>Q12</td>
<td>4x4</td>
<td>3x3</td>
<td>3x3</td>
</tr>
</tbody>
</table>

Minimum integration order

In order to pass the patch test, shape function should be able to represent constant strain.

\[ \varepsilon = \text{const} \]

\[
U_e = \frac{1}{2} \int_V \{ \varepsilon \}^T \{ \sigma \} dV_e = \frac{1}{2} \{ \varepsilon \} \{ \sigma \} \int_{V_e} dV_e = \frac{1}{2} U_o \int_{V_e} dV_e
\]

\[
U_e = \frac{1}{2} U_o \int_{V_e} \sqrt{J} d\eta d\xi
\]

Minimum integration order should be able to compute exactly the volume of element.
Three dimensions

\[ I = \iiint_{-1}^{1} \phi(\xi, \eta, \zeta) d\xi d\eta d\zeta \approx \sum_i \sum_j \sum_k W_i W_j W_k \phi(\xi_i, \eta_j, \zeta_k) \]

Computational cost of 3x3x3 Gauss quadrature for the brick element:

27 (Gauss points) x 24x24 = 15,552 (function evaluations per element)

Next class

Validity of isoparametric element

- Ability to represent rigid body motions
- Generalized Iso-P formulation: GIF
- Graded element and homogeneous element