Approximating Sensitivity of Failure Probability in Reliability-Based Design Optimization

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Motivation - Importance of Accurate Sensitivity

In gradient-based optimization, regardless of what algorithm is used, the sensitivity analysis (gradient information) is essential:

• **Where to go**
  The gradients of objective and constraint functions determine the step direction at each iteration during the optimization

• **When to stop**
  The gradients are used to judge whether a (local) optimum has been reached, because they are important components of the Karush-Kuhn-Tucker (KKT) optimality conditions.
Motivation - Importance of Accurate Sensitivity

Plot of the general procedure of a gradient-based optimization

- $\nabla_x f$: true gradient
- $\nabla_x g$: approximate gradient
- $g(x) < 0$: constraint function
- $f(x)$: objective function
- $x$: design variables

<Design Space>
Outline

• Derivation of the parameter sensitivity of failure probability
• Review of FORM-based approximation
• The proposed approximation method
• Numerical assessment and examples
• Conclusion
The analytical expression for the sensitivity of failure probability w.r.t design parameters is a surface integral

\[ \Delta \delta P_f = -\int_{\Delta D} \phi_n(u) d\mathbf{u} \]

Evolvement of limit state surface

\[ \nabla_u G \delta \mathbf{u} + \nabla_x G \delta \mathbf{x} = 0 \]

\[ \Delta D = \int_S \delta \mathbf{u}^T n dS \]

Parameter sensitivity of \( P_f \) [1, 2]

\[ \nabla_x P_f = -\int_S \frac{\phi_n(u)}{\| \nabla_u G \|} \nabla_x G dS \]

FORM-Based Approximation

Analytical expression:

\[ \nabla_x P_f = -\int_S \frac{\varphi_n(u)}{\| \nabla_u G \|} \nabla_x G dS \]

Hohenbichler and Rackwitz [1]:

\[ \nabla_x P_{f,1} = -\frac{\varphi(||u^*||)}{\| \nabla_u G(u^*, x) \|} \nabla_x G(u^*, x) \]

Especially when the design parameter \( x \) influence the curvature of limit state surface, the FORM-based approximation is totally blind to that.

FORM-based approximation is exact when the limit state function is linear and can be quite inaccurate for nonlinear limit state functions

Proposed Segmental Multi-point Linearization (SML)

We can simplify the surface integration by approximating it on a piecewise linear fitting of the limit state surface.

On each hyperplane segment of the multi-point linear fitting:

\[
\nabla_x P_f^i = \int_{\bar{S}_i} -\frac{\varphi_n(u)}{||\nabla_u G||} \nabla_x Gd\bar{S}_i
\]

\[
= -\frac{\varphi(b_i)\int_{\bar{S}_i} \varphi_{n-1}(\hat{u}')d\bar{S}_i}{||\nabla_u \bar{G}^i||} \nabla_x \bar{G}^i
\]

The above derivation is used to construct the weighted sum approximation of \(\nabla_x P_f\):

\[
\nabla_x P_f = \sum_{i=1}^{p} \nabla_x P_f^i = \sum_{i=1}^{p} W_i \nabla_x \bar{G}^i
\]

SML – Different Fitting Schemes

The essence of SML method is a proper multi-segmental linear fitting of the limit state surface.

<table>
<thead>
<tr>
<th>Tangent Fitting (TF)</th>
<th>Step Fitting (SF)</th>
<th>Orthogonal Fitting (OF)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pros:</strong> accurate, can be refined</td>
<td><strong>Pros:</strong> simple geometry, weights easy to compute, can be refined</td>
<td><strong>Pros:</strong> simple geometry, weights easy to compute</td>
</tr>
<tr>
<td><strong>Cons:</strong> complicated geometry, weights hard to compute</td>
<td><strong>Cons:</strong> not good for large curvature</td>
<td><strong>Cons:</strong> cannot be refined</td>
</tr>
</tbody>
</table>

Remarks: The SML also provides approximations of the failure probability at no additional cost and the approximations are often more accurate than the approximations by FORM.
Benefits of the SML Methods

The SML method finds a good balance between computational cost and accuracy comparing to other existing methods.

Qualitative comparison of computational cost vs. accuracy

- Efficient
  - FORM-based approximation
  - MCS-based approximation
- Inefficient
  - SML
  - SML-OF

Computational Cost

Accuracy

# of fitting points
All fitting schemes of SML show improvement in accuracy comparing to FORM-based approximations of $\nabla_x P_f$, as well as $P_f (\beta)$.

Consider a limit state function defined directly in the standard normal random space:

$$G(u_1, u_2, x) = 3 - u_2 - xu_1^2$$
In optimization, the direction of an approximate gradient is the main concern about its accuracy, and the OF scheme shows its power

\[ g(v, x) = x_3 - v_3 - x_2v_2^2 - x_1v_1^2 \]

\( v \) are original random variables subject standard normal marginal distribution with correlations:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0.2 \\
0 & 0.2 & 1
\end{bmatrix}
\]

The measure \( \alpha \) is the relative angle between the approximate gradient and the “actual” gradient computed by MCS:

\[
\cos^{-1}\left( \left( \nabla u P_f, (\nabla u P_f)_{MCS} \right) \right)
\]
Numerical Examples – RBDO of Concrete Girder

The SML (OF) shows a good agreement with the result obtained using a MCS-based RBDO algorithm

Roysset, Polak [1]:

D.V.=[As, b, hf, bw, hw, Av, S1, S2, S3]
R.V.=[fy, f’c, PD, ML, W, PS1, PS2, PS3]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Design by SAA [1]</th>
<th>Design by SML</th>
</tr>
</thead>
<tbody>
<tr>
<td>As</td>
<td>0.008954</td>
<td>0.008916</td>
</tr>
<tr>
<td>b</td>
<td>0.384</td>
<td>0.379</td>
</tr>
<tr>
<td>hf</td>
<td>0.411</td>
<td>0.415</td>
</tr>
<tr>
<td>bw</td>
<td>0.197</td>
<td>0.196</td>
</tr>
<tr>
<td>hw</td>
<td>0.789</td>
<td>0.785</td>
</tr>
<tr>
<td>Av</td>
<td>0.0001685</td>
<td>0.0001555</td>
</tr>
<tr>
<td>S1</td>
<td>0.535</td>
<td>0.457</td>
</tr>
<tr>
<td>S2</td>
<td>0.230</td>
<td>0.204</td>
</tr>
<tr>
<td>S3</td>
<td>0.143</td>
<td>0.129</td>
</tr>
<tr>
<td>Cost (Obj.)</td>
<td>12.696</td>
<td>12.660</td>
</tr>
<tr>
<td># of Func. Eval.</td>
<td>&gt;1,860,000</td>
<td>&lt;2,304</td>
</tr>
</tbody>
</table>

Numerical Examples – Application to RBTO

The proposed method shows its advantage when applied to Reliability-Based Topology Optimization problems.

\[ \min \text{Volume} (L^T A): P(\text{compliance} (F^T d) > \text{threshold}) < \text{target } P_f \]

Deterministic topology optimization (mean value)

RBTO using FORM-based method

RBTO using SML (OF)

Random forces \( \sim N(7,3) \)
Numerical Examples – Application to RBTO

Reliability-Based Topology Optimization of a 3D tower crane

Young’s Modulus is also a random variable, which is subjected to lognormal distribution.

Conclusions

• The work proposes a method that can approximate the parameter sensitivity of the failure probability in a good accuracy without requiring high computational cost

• The proposed method can be applied to a variety of reliability-based design optimization problems, and whenever the sensitivity of failure probability needs to be computed with good accuracy

• The method have many variations that can provide approximations with different accuracy depending on how the limit state surface is fitted

• The method is a general theory and framework, which has the potential to be further improved in the future by developing new fitting schemes
The approximations of $\nabla_x P_f$ and $P_f (\beta')$ are at the same level of accuracy. Some times $P_f$ is improved using other reliability method, but little a attention has been paid to the accuracy of $\nabla_x P_f$.