On Optimization of Shape and Topology

Cameron Talischi, Glaucio H. Paulino

Department of Civil and Environmental Engineering
University of Illinois at Urbana-Champaign

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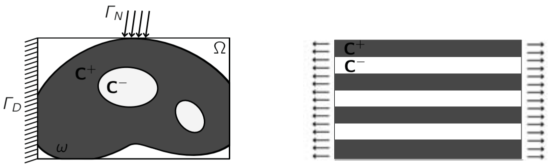
Introduction:

- The goal of optimal shape design is to find the most efficient shape of a physical system
- The response is captured by the solution $\mathbf{u}_\omega$ to a boundary value problem that in turn depends on the given shape ω

$$\inf_{\omega\subseteq\Omega} J(\omega,\mathbf{u}_\omega) \quad \text{where} \quad \mathcal{B}(\mathbf{u}_\omega,\mathbf{v};\omega)=\ell(\mathbf{v}),\ \forall\mathbf{v}\in\mathcal{V}$$



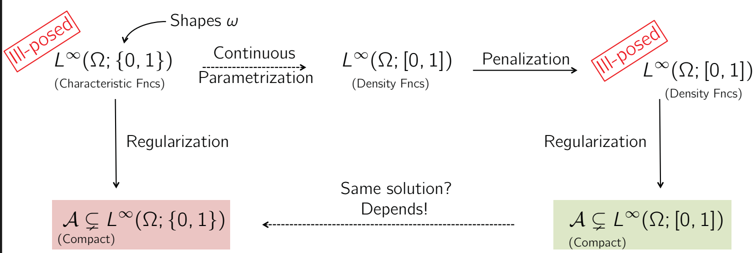
$$\mathcal{B}(\mathbf{u},\mathbf{v};\omega)=\int_\Omega \nabla\mathbf{u}:\left[\chi_\omega\mathbf{C}^+ + (1-\chi_\omega)\,\mathbf{C}^-\right]:\nabla\mathbf{v}\mathrm{d}\mathbf{x}, \qquad \ell(\mathbf{v})=\int_{\Gamma_N}\mathbf{t}\cdot\mathbf{v}\mathrm{d}s$$

Restriction setting:

- If $\chi_n,\hat{\chi}\in L^\infty(\Omega;[0,1])$ such that $\chi_n\to\hat{\chi}$ in $L^1(\Omega)$, then, up to a subsequence, the associated state solutions also converge, i.e., $\mathbf{u}_{\chi_n}\to\mathbf{u}_{\hat{\chi}}$ in $H^1(\Omega;\mathbb{R}^d)$
- It follows that compactness in $L^1(\Omega)$ topology is a sufficient condition for existence of solutions
- A well-known example is the space of shapes with bounded perimeter:

$$\mathcal{A}=\left\{\chi\in BV(\Omega;\{0,1\}) : \int_\Omega|\nabla\chi|\,\mathrm{d}\mathbf{x}\le\overline{P}\right\}$$

Continuous parametrization:



Optimization problem:

Composite objective: $\min_{\rho\in\mathcal{A}} F(\rho) := J(\rho)+R(\rho)$

Performance functional: $J(\rho)=\int_{\Gamma_N}\mathbf{t}\cdot\mathbf{u}_\rho\mathrm{d}s+\lambda\int_\Omega\rho\mathrm{d}\mathbf{x}$

Regularizer: $R(\rho)=\frac{\beta}{2}\int_\Omega|\nabla\rho|^2\,\mathrm{d}\mathbf{x}\equiv\frac{1}{2}\langle\rho,\mathcal{R}\rho\rangle,\quad \mathcal{R}=-\beta\Delta$

Admissible densities: $\mathcal{A}=\left\{\rho\in H^1(\Omega):0\le\rho\le1\right\}$

State equation: $\int_\Omega\nabla\mathbf{u}_\rho:\mathbf{C}_\rho:\nabla\mathbf{v}\mathrm{d}\mathbf{x}=\int_{\Gamma_N}\mathbf{t}\cdot\mathbf{v}\mathrm{d}s,\ \forall\mathbf{v}\in\mathcal{V}$

$\mathbf{C}_\rho=\rho^p\mathbf{C}^+ + (1-\rho^p)\,\mathbf{C}^-$

Forward-backward splitting algorithm:

- We consider an optimization algorithm of the form:

$$\rho_{n+1}=\operatorname*{argmin}_{\rho\in\mathcal{A}}\frac{1}{2\tau_n}\left\|\rho-\left[\rho_n-\tau_nJ'(\rho_n)\right]\right\|^2+R(\rho)$$

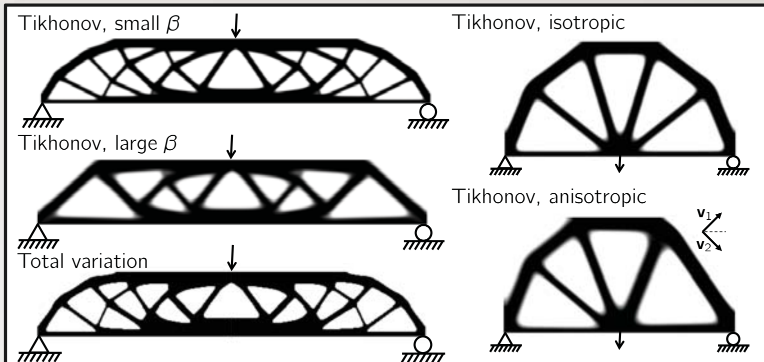
The intuition is that the next iterate ρ_{n+1} is close to the gradient descent update on J, i.e., $\rho_n-\tau_nJ'(\rho_n)$, while minimizing the regularizer $R(\rho)$

- Given constants $\tau_0>0$ and $0<\sigma<1$, the step size parameter is set to be

$$\tau_n=\sigma^{k_n}\tau_0$$

where k_n is the smallest non-negative integer such that τ_n satisfies

$$F(\rho_n)-F(\rho_{n+1})\ge\frac{1}{2\tau_n}\|\rho_n-\rho_{n+1}\|^2$$



Improving convergence:

- We consider the following generalization:

$$\rho_{n+1}=\operatorname*{argmin}_{\rho\in\mathcal{A}} J(\rho_n)+\langle J'(\rho_n),\rho-\rho_n\rangle+\frac{1}{2\tau_n}\langle\rho-\rho_n,\mathcal{H}_n(\rho-\rho_n)\rangle+R(\rho)$$

where $\mathcal{H}_n$ is a bounded linear positive-definite operator

- The reciprocal approximation of compliance is its Taylor expansion in the intermediate field ρ^{-1}

$$J_{\mathrm{rec}}(\rho;\rho_n)=J(\rho_n)+\langle J'(\rho_n),\rho-\rho_n\rangle+\frac{1}{2}\left\langle\rho-\rho_n,\frac{2E(\rho_n)}{\rho}(\rho-\rho_n)\right\rangle$$

where $E(\rho)\equiv p\rho^{p-1}\left[\nabla\mathbf{u}_\rho:(\mathbf{C}^+-\mathbf{C}^-):\nabla\mathbf{u}_\rho\right]$ is the gradient of compliance.

- We embed the same type of approximation into our quadratic model by setting

$$\mathcal{H}_n=J''_{\mathrm{rec}}(\rho_n;\rho_n)=\frac{2E(\rho_n)}{\rho_n}\mathcal{I}$$

Performance of the algorithm:

Algorithm	$\mathcal{H}_n$	τ_0	# Iter.	# BT	F	OC
GP	–	0.25	1000*	0	210.74	1.36e-4
GP	–	0.5	568	79	210.68	8.94e-5
FBS	Identity	1	316	0	210.97	9.94e-5
FBS	Identity	2	215	154	210.91	9.81e-5
FBS	Reciprocal	1	186	0	211.03	9.36e-5
FBS	Reciprocal	2	91	39	211.00	9.75e-5
TM-FBS	Identity	1	330	0	210.95	9.97e-5
TM-FBS	Identity	2	151	78	210.94	5.90e-5
TM-FBS	Reciprocal	1	179	0	211.03	9.45e-5
TM-FBS	Reciprocal	2	85	34	211.00	8.07e-5
MMA	–	–	1000*	–	213.39	1.91e-4