Single-Loop System Reliability-Based Design Optimization Using Matrix-Based System Reliability Method: Theory and Applications

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This paper proposes a single-loop system reliability-based design optimization (SRBDO) approach using the recently developed matrix-based system reliability (MSR) method. A single-loop method was employed to eliminate the inner-loop of SRBDO that evaluates probabilistic constraints. The MSR method enables us to compute the system failure probability and its parameter sensitivities efficiently and accurately through convenient matrix calculations. The SRBDO/MSR approach proposed in this paper is applicable to general systems including series, parallel, cut-set, and link-set system events. After a brief overview on SRBDO algorithms and the MSR method, the SRBDO/MSR approach is introduced and demonstrated by three numerical examples. The first example deals with the optimal design of a combustion engine, in which the failure is described as a series system event. In the second example, the cross-sectional areas of the members of a statically indeterminate truss structure are determined for minimum total weight with a constraint on the probability of collapse. In the third example, the redistribution of the loads caused by member failures is considered for the truss system in the second example. The results based on different optimization approaches are compared for further investigation. Monte Carlo simulation is performed in each example to confirm the accuracy of the system failure probability computed by the MSR method.
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1 Introduction

The main objective of design optimization is to obtain the values of design variables that minimize or maximize the objective function(s) of interest while satisfying given design constraints. If design optimization is performed in a deterministic manner, that is, uncertainties are not taken into account during the optimization, the resultant optimal design may have unquantified risk of violating the given constraints. Various reliability-based design optimization (RBDO) methods have been developed to achieve optimal designs with acceptable failure probabilities (see Refs [1,2] for a state-of-the-art review of RBDO methods and recent applications to civil and aerospace structural systems). During RBDO, the probability of violating given constraint(s), namely, the failure probability, is often computed by reliability analysis methods such as first-order reliability method (FORM), second-order reliability method (SORM), or response surface method.

Traditionally, RBDO has been performed by use of a nested or "double-loop" approach, in which each step of the iterations for design optimization involves another loop of iterations for reliability analysis. For example, the reliability index approach (RIA) [3] and performance measure approach (PMA) [4,5] employ FORM to perform the reliability analysis, which requires nonlinear constrained optimization (for a review on FORM, see Ref. [6]). If the constraints are active, the two approaches yield the same results. However, it is known that PMA is generally more efficient and stable than RIA [4,5]. The double-loop computation can be prohibitive if the function evaluation cost is expensive because the inner-loop often involves iterative reliability analysis to search for the most probable point (MPP) [7–9]. As an effort to reduce the computational burden of RBDO, many approximate RBDO approaches have been developed to decouple the double-loop problem [9–21]. For example, a single-loop approach [21] was proposed by using the Karush–Kuhn–Tucker (KKT) optimality condition to approximate the solution of the inner-loop optimization. As a result, the inner-loop is replaced by a deterministic constraint, which transforms a double-loop RBDO problem into an equivalent single-loop optimization problem.

When multiple failure modes need to be considered as the constraints of a design optimization, RBDO is often formulated such that the optimal structure satisfies each failure mode with predetermined probabilities. This approach is termed as "component reliability-based design optimization (CRBDO)" in this paper. In some cases, however, the failure event is better described by a system event, i.e., a logical (or Boolean) function of multiple failure modes. In this case, the probabilistic constraint should be given for the system event, not on individual component failure modes. This approach is called "system reliability-based design optimization (SRBDO)." The SRBDO requires system reliability analysis, which is not trivial, especially for systems with statistically dependent component events, or for events that are not series or parallel systems. Theoretical bounding formulas are applicable to parallel and series systems only (see Ref. [22] for a review), and it is inconvenient to deal with probability bounds during

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RBDO. Various sampling methods are available, but they may render SRBDO inefficient in practice. Song and Kang [23] recently developed a matrix-based system reliability (MSR) method that computes the system reliability by convenient matrix-based framework. The MSR method is applicable to general system events including series, parallel, cut-set, and link-set systems, and can account for statistical dependence between component events. It also provides parameter sensitivities of the failure probability for general system events, which facilitates efficient RBDO.

This paper aims to overcome aforementioned challenges in SRBDO by integrating a single-loop SRBDO approach with the MSR method (SRBDO/MSR). After an overview of existing RBDO formulations and the MSR method, the proposed SRBDO/MSR procedure is explained. The MSR method is further developed for integration with a single-loop SRBDO approach. The proposed SRBDO/MSR approach is demonstrated by three numerical examples.

2 System Reliability-Based Design Optimization

2.1 Component Reliability-Based Design Optimization. In general, RBDO problems are formulated as follows:

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}) \\ &\text{s.t.}\quad P[g_i(\mathbf{d},\mathbf{X}) \le 0] \le P_i^t, \quad i=1,\ldots,n \\ &\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \quad \boldsymbol{\mu}_{\mathbf{X}}^L \le \boldsymbol{\mu}_{\mathbf{X}} \le \boldsymbol{\mu}_{\mathbf{X}}^U \end{aligned} \tag{1}$$

where $\mathbf{d} \in \Re^k$ is the vector of deterministic design variables; $\mathbf{X} \in \Re^m$ is the vector of random variables; $\boldsymbol{\mu}_{\mathbf{X}}$ is the vector of the means of $\mathbf{X}$; $f(\cdot)$ is the objective function; $g_i(\cdot)$, $i=1,\ldots,n$ is the ith limit-state function indicating the occurrence of the failure by $g_i(\cdot)\le 0$; P_i^t is the constraint on the probability of the ith limit-state; $\mathbf{d}^L$ and $\mathbf{d}^U$ are the lower/upper bounds on $\mathbf{d}$; $\boldsymbol{\mu}_{\mathbf{X}}^L$ and $\boldsymbol{\mu}_{\mathbf{X}}^U$ are the lower/upper bounds on $\boldsymbol{\mu}_{\mathbf{X}}$ (for simplicity, these boundary values will be omitted in the following RBDO formulations in the paper); and n, k, and m are the numbers of constraints, deterministic design variables, and random variables, respectively. The probabilistic constraint in Eq. (1) can be given alternatively by use of the cumulative distribution function (CDF) of the limit-state function, that is,

$$P[g_i(\mathbf{d},\mathbf{X}) \le 0] = F_{g_i}(0) \le \Phi(-\beta_i^t) \tag{2}$$

where $F_{g_i}(\cdot)$ denotes the CDF of $g_i(\cdot)$; $\Phi(\cdot)$ is the CDF of the standard normal distribution; and β_i^t is the target reliability index. First-order reliability method [6] is widely employed to compute failure probability in Eq. (2). In all the numerical examples of this paper, FORM is used for component-level reliability analysis.

This RBDO problem has two nested optimization loops: the outer-loop for design optimization and the inner-loop for reliability analysis. One of the double-loop approaches commonly used for RBDO is the reliability index approach (RIA) [3], which uses the formulation

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}) \\ &\text{s.t.}\quad \beta_i = -\Phi^{-1}[F_{g_i}(0)] \ge \beta_i^t, \quad i=1,\ldots,n \end{aligned} \tag{3}$$

where β_i is the distance from the origin of the space of standard normal random variables $\mathbf{U}=\mathbf{U}(\mathbf{X})$ to the nearest point on the limit-state surface $G_i(\mathbf{d},\mathbf{U})=0$, in which $G_i(\cdot)$ is the limit-state function $g_i(\cdot)$ determined in terms of $\mathbf{U}$, that is, $g_i(\mathbf{d},\mathbf{X}) = G_i(\mathbf{d},\mathbf{U}(\mathbf{X}))$. This distance β_i is termed as the "reliability index." The nearest point on the limit-state surface, often termed as "design point" or "most probable failure point (MPP)" is identified by solving a nonlinear constrained optimization [6]

$$\begin{aligned} &\mathbf{U}_i^* = \arg\min_{\mathbf{U}} \|\mathbf{U}\| \\ &\text{s.t.}\quad G_i(\mathbf{d},\mathbf{U}) = 0 \end{aligned} \tag{4}$$

where $\mathbf{U}_i^*$ is the MPP of the ith limit-state function, and "arg min" denotes the argument of the minimum of a function.

The RIA formulation in Eq. (3) can be inefficient if the constraints are inactive. Moreover, the algorithm may not provide an optimal design solution if the failure events $G_i(\mathbf{d},\mathbf{U})\le 0$ never occur in the given feasible domain. To overcome these issues, Tu et al. [4] proposed the performance measure approach (PMA), in which the probabilistic constraint is described in terms of "performance function," which is defined as the quantile of the limit-state function $g_i(\cdot)$ at the target failure probability $\Phi(-\beta_i^t)$. It is thus formulated as

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}} f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}}) \\ &\text{s.t.}\quad g_{p_i} = F_{g_i}^{-1}[\Phi(-\beta_i^t)] \ge 0, \quad i=1,\ldots,n \end{aligned} \tag{5}$$

where g_{p_i} is the performance function. The constraint in Eq. (5) implies that $F_{g_i}(g_{p_i})=\Phi(-\beta_i^t)$ is greater than $F_{g_i}(0)=\Phi(-\beta_i)$, so it is equivalent to the constraint in Eq. (3), $\beta_i \ge \beta_i^t$. The performance function can be obtained by solving a constrained optimization problem [4,5,24]

$$\begin{aligned} &g_{p_i} = \min_{\mathbf{U}} G_i(\mathbf{d},\mathbf{U}) \\ &\text{s.t.}\quad \|\mathbf{U}\| = \beta_i^t \end{aligned} \tag{6}$$

To improve efficiency of these double-loop RBDOs, several single-loop RBDO approaches have been developed [9–21]. For example, a sequential optimization and reliability assessment (SORA) method was recently proposed [11]. Its main idea is to decouple the outer-loop optimization from reliability analysis. Using the information from previous design iteration, the boundaries of the constraints are shifted to the feasible direction and the design point is updated accordingly. Additionally, the safety-factor approach [13,14], one of the single-loop approaches, was developed by using the approximate equivalent deterministic constraint to convert the double-loop into single-loop problem. The efficiency of double-loop approach can be enhanced by some efficiency strategies such as the enriched performance measure approach (PMA+) [7]. It was reported that with such efficiency strategies, double-loop approach can be more efficient than single-loop approach [7,8].

Recently, Liang et al. [21] proposed a single-loop RBDO by approximating the result of the nonlinear constrained optimization in Eq. (6) by solving the system equations that describe the KKT condition

$$\begin{aligned} &\nabla_{\mathbf{U}} G_i(\mathbf{d},\mathbf{U}) + \lambda \cdot \nabla_{\mathbf{U}}(\|\mathbf{U}\| - \beta_i^t) = \mathbf{0} \\ &\|\mathbf{U}\| - \beta_i^t = 0 \end{aligned} \tag{7}$$

in which λ denotes a Lagrange multiplier. Next, the "negative normalized gradient vector" [6] of the limit-state function at the solution of Eq. (6) is approximately obtained by evaluating it at the solution of Eq. (7), $\mathbf{U}=\tilde{\mathbf{U}}_i$, that is

$$\hat{\boldsymbol{\alpha}}_i^t \cong \left(-\frac{\nabla_{\mathbf{X}} g_i(\mathbf{d},\mathbf{X}(\mathbf{U}))}{\|\nabla_{\mathbf{X}} g_i(\mathbf{d},\mathbf{X}(\mathbf{U}))\|} \mathbf{J}_{\mathbf{X},\mathbf{U}} \right)_{\mathbf{U}=\tilde{\mathbf{U}}_i} \tag{8}$$

where $\mathbf{J}_{\mathbf{X},\mathbf{U}}$ is the Jacobian of the $\mathbf{X}=\mathbf{X}(\mathbf{U})$ transformation. The solution of Eq. (6) is then approximated by scaling this unit vector by the target reliability index, i.e.,

$$\mathbf{U}_i^t \cong \beta_i^t \hat{\boldsymbol{\alpha}}_i^t \tag{9}$$

The performance function is then approximated by evaluating the limit-state function at $\mathbf{U}_i^t$. As a result, the RBDO is formulated as

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X}) \\ &\text{s.t.} \quad g_{p_i} \cong g_i(\mathbf{d},\mathbf{X}(\mathbf{U}_i^t)) \geq 0, \quad i=1,\ldots,n \end{aligned} \tag{10}$$

In summary, the inner-loop of the PMA RBDO is replaced by the approximate noniterative procedures shown in Eqs. (7)–(9). This single-loop approach was reported to have the accuracy comparable with the double-loop approach and the efficiency almost equivalent to deterministic optimization [21]. This study aims to improve this single-loop RBDO approach when *system* reliability analysis is needed for failure probability calculations.

2.2 System Reliability-Based Design Optimization. In the case when the failure event in the design constraint is better described by a system event, i.e., a logical (Boolean) function of multiple component events, the RBDO requires a system reliability analysis. This system reliability-based design optimization (SRBDO) can be formulated as

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X}) \\ &\text{s.t.} \quad P_{\text{sys}} = P(E_{\text{sys}}) = P[\bigcup_k \bigcap_{i\in C_k} g_i(\mathbf{d},\mathbf{X}) \leq 0] \leq P_{\text{sys}}^t \end{aligned} \tag{11}$$

where P_{sys} is the system failure probability; E_{sys} is the system failure event; C_k is the index set of the components in the kth cut set; and P_{sys}^t is the target system failure probability. Any type of system event may be used during SRBDO but, for illustration purposes, Eq. (11) shows a cut-set system formulation that can represent series, parallel, and cut-set systems. Royset et al. [12] proposed a decouple procedure for RBDO of series systems. The target system reliability is achieved by adjusting the target component reliabilities heuristically.

An SRBDO approach was proposed for series system problems in Ref. [25]. In this approach, the failure probability of a series system is approximated as the sum of the component failure probabilities, i.e.,

$$P_{\text{sys}} = P\left[\bigcup_{i=1}^{n} g_i(\mathbf{d},\mathbf{X}) \leq 0\right] \cong \min\left(1, \sum_{i=1}^{n} P_i\right) \tag{12}$$

Then, SRBDO problems are formulated as

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X},P_1^t,\ldots,P_n^t} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X}) \\ &\text{s.t.} \quad P[g_i(\mathbf{d},\mathbf{X}) \leq 0] \leq P_i^t, \quad i=1,\ldots,n \\ &P_{\text{sys}} \cong \min\left(1, \sum_{i=1}^{n} P_i^t\right) \leq P_{\text{sys}}^t \end{aligned} \tag{13}$$

Note that the constraints on the component probabilities, P_i^ts are used as design variables. This approach can significantly overestimate the system risk because the approximation in Eq. (12) provides a fairly conservative upper bound (see Ref. [22] for a review on system reliability bounding formulas). Moreover, this approach cannot account for the effect of the statistical dependence between component events, which is caused by common random variables or statistical correlation between random variables.

A single-loop SRBDO approach was recently proposed for series systems by Liang et al. [26]. This approach also uses P_i^ts as design variables. The inner-loop is eliminated by approximating the design points by KKT conditions as explained above. The system failure probability is approximated as the upper bound by the bi-component theoretical bounding formula [27]. As a result, the single-loop SRBDO is formulated as

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X},P_1^t,\ldots,P_n^t} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X}) \\ &\text{s.t.} \quad g_i(\mathbf{d},\mathbf{X}(\mathbf{U}_i^t)) \geq 0, \quad i=1,\ldots,n \\ &P_{\text{sys}} \cong \sum_{i=1}^{n} P_i^t - \sum_{i=2}^{n} \max_{j<i} P_{ij}^t \leq P_{\text{sys}}^t \end{aligned} \tag{14}$$

in which $\mathbf{U}_i^t$ is obtained by Eqs. (7)–(9); and P_{ij}^t is the joint failure probability of the ith and jth constraints, computed by a numerical integration based on P_i^t, P_j^t, and the inner-product of approximated negative normalized gradient vectors [26]. Despite its improved accuracy in estimating the system failure probability by using a higher-order bounding formula, it still overestimates the system failure probability and is not applicable to nonseries system events for which general theoretical bounding formulas are not available.

In this paper, we propose to use the recently developed matrix-based system reliability method to compute P_{sys} in the single-loop SRBDO shown in Eq. (14). The method enables us to compute P_{sys} of general system events including series, parallel, cut-set, and link-set systems efficiently and accurately during SRBDO. The sensitivity of P_{sys} with respect to design variables further facilitates the use of gradient-based optimization algorithms.

3 System Reliability-Based Design Optimization Using MSR Method

3.1 Matrix-Based System Reliability Method. Although system reliability analysis is a well established research area, it is still challenging to compute the probability of a general system event and its parameter sensitivity, especially when component events are statistically dependent. Song and Der Kiureghian [22] introduced a method to compute the bounds on the probability of a general system event by linear programming (LP). This "LP bounds" method subdivides the sample space of component events into the mutually exclusive and collectively exhaustive events (termed as basic MECE events), and the probability of any event is described by use of vectors representing the probabilities of basic MECE events. Then, its upper and lower bounds are obtained by solving the LP problems subjected to the constraints derived from given information such as component probabilities and statistical dependence. This matrix-based framework of system reliability analysis enables obtaining the narrowest possible bounds on the probability of any general system and the parameter sensitivities of the bounds [28] as well.

Song and Kang [23] recently proposed the MSR method to compute the probability of general system events by use of simple matrix calculations instead of solving LP. Consider a system event with n components each of which has two distinct states, e.g., "failure" and "safe." Then, the sample space can be subdivided into $N=2^n$ basic MECE events denoted by e_j, $j=1,\ldots,N$. Then any system event can be presented by an "event" vector $\mathbf{c}$ whose jth element is 1 if e_j belongs to the system event and 0 otherwise. Let $p_i=P(e_j)$, $j=1,\ldots,N$, denote the probability of e_j. Because e_js are mutually exclusive to each other, the probability of system event, P_{sys} is simply the sum of the probability of e_js that belong to the system event E_{sys}. Therefore, the system probability is computed by the inner-product of the two vectors

$$P_{\text{sys}} = \sum_{j:e_j\subseteq E_{\text{sys}}} p_j = \mathbf{c}^{\text{T}}\mathbf{p} \tag{15}$$

where $\mathbf{p}$ is the "probability" vector that contains p_js, $j=1,\ldots,N$. Both $\mathbf{c}$ and $\mathbf{p}$ are column vectors in this paper and can be constructed efficiently using matrix-based procedures proposed in Ref. [23].

When component events are statistically dependent, the construction of $\mathbf{p}$ requires system reliability analysis for each element. This challenge can be overcome by achieving conditional independence between component events given outcomes of a few random variables representing the sources of "environment dependence" or "common source effects." For example, during a risk analysis of a transportation network based on bridge failure probabilities, the uncertain magnitude of earthquake was considered as a random variable representing the common source effect [29]. Let $\mathbf{S}$ denote the vector of such random variables, named "common source random variables" (CSRV). By the total probability theorem, the system failure probability can be then computed as

$$P_{\text{sys}} = \int_{\mathbf{s}} P(E_{\text{sys}}|\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} = \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}} \mathbf{p}(\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} \tag{16}$$

where $P(E_{\text{sys}}|\mathbf{s})$ is the conditional probability of the system event given an outcome of CSRV, $\mathbf{S}=\mathbf{s}$; $f_{\mathbf{S}}(\mathbf{s})$ is the joint probability density function (PDF) of $\mathbf{S}$; and $\mathbf{p}(\mathbf{s})$ is the conditional probability vector given $\mathbf{S}=\mathbf{s}$, which can be constructed efficiently by the proposed matrix-based procedure employing conditional probabilities of component events given $\mathbf{S}=\mathbf{s}$, i.e., $P_i(\mathbf{s})=P(E_i|\mathbf{S}=\mathbf{s})$ instead of the marginal probabilities $P_i=P(E_i)$.

The approach in Eq. (16) can be used even in the case when the CSRVs are not explicitly identified. One way to identify such implicit common source effect as CSRVs is to fit the correlation coefficient matrix of random variables representing component events such as safety margin (or factor) with a special correlation matrix model that allows such an identification. For example, Song and Kang [23] generalized Dunnett–Sobel (DS) class correlation matrix [30] to identify CSRVs. Consider correlated standard normal random variables Z_i, $i=1,\ldots,n$. Their correlation matrix can be fit with the following generalized DS model through an optimization:

$$Z_i = \left(1 - \sum_{k=1}^{m} r_{ik}^2\right)^{0.5} Y_i + \sum_{k=1}^{m} r_{ik} S_k \quad \text{for} \quad i=1,\ldots,n \tag{17}$$

in which Y_i, $i=1,\ldots,n$ and S_k, $k=1,\ldots,m$ are uncorrelated standard normal random variables; and r_{ik}s are the coefficients of the generalized DS model that determine the correlation coefficient between Z_i and Z_j as $\rho_{ij}=\sum_{k=1}^{m}(r_{ik}.r_{jk})$ for $i\neq j$. Note that Z_i and Z_j are conditionally independent of each other given the outcome of CSRVs S_k, $k=1,\ldots,m$. The MSR method is demonstrated by an illustrative example in the Appendix of this paper.

3.2 Parameter Sensitivity of System Failure Probability. The MSR method enables us to compute the parameter sensitivity of the probability of a general system event. First, when the component events are statistically independent, the sensitivity of the system failure probability with respect to a parameter θ is computed as

$$\frac{\partial P_{\text{sys}}}{\partial \theta} = \mathbf{c}^{\mathrm{T}} \frac{\partial \mathbf{p}}{\partial \theta} \tag{18}$$

The separation of the system event description ($\mathbf{c}$) and the probabilities ($\mathbf{p}$) in the MSR framework allows us to compute the parameter sensitivity for general system events in a uniform manner. The sensitivity of $\mathbf{p}$ in Eq. (18) can be computed by the following matrix-based procedure [23]:

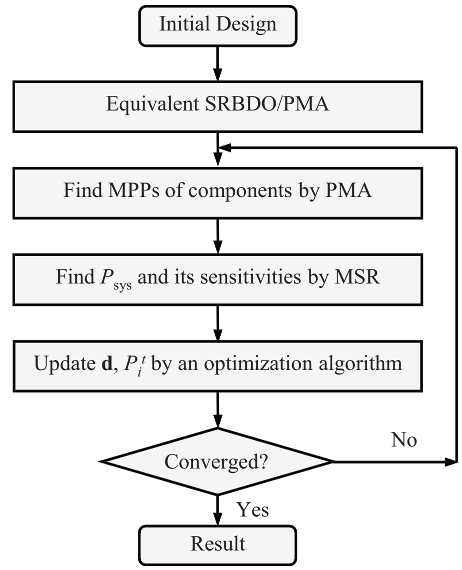

Fig. 1 Flowchart of the proposed SRBDO/MSR algorithm

$$\frac{\partial \mathbf{p}}{\partial \theta} = [\mathbf{p}^{\langle 1\rangle}\mathbf{p}^{\langle 2\rangle}\ldots\mathbf{p}^{\langle n\rangle}]\frac{\partial \mathbf{P}}{\partial \theta} = \hat{\mathbf{P}}\frac{\partial \mathbf{P}}{\partial \theta} \tag{19}$$

where $\mathbf{P}=[P_1P_2\cdots P_n]^{\mathrm{T}}$ in which P_i is the probability of the ith component event; and $\mathbf{p}^{\langle j\rangle}$, $j=1,\ldots,n$ is the probability vector constructed by the matrix-based procedure developed for $\mathbf{p}$ except that the probabilities of the jth component event and its complementary event are replaced by 1 and −1, respectively, during the construction. In summary, the MSR framework allows us to compute the system-level parameter sensitivities by use of component probabilities and their parameter sensitivities.

When the components are statistically dependent, the parameter sensitivity is computed as

$$\frac{\partial P_{\text{sys}}}{\partial \theta} = \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}} \frac{\partial \mathbf{p}(\mathbf{s})}{\partial \theta} f_S(\mathbf{s}) d\mathbf{s} \tag{20}$$

in which the sensitivity in the integral is constructed by the procedure in Eq. (19) except that the conditional probability of the component events given $\mathbf{S}=\mathbf{s}$, i.e.,

$$P_i(\mathbf{s}) = P(\beta_i - Z_i \leq 0|\mathbf{S}=\mathbf{s}), \quad i=1,\ldots,n \tag{21}$$

is used instead of P_i. Substituting Eq. (17) into Eq. (21), the conditional probability is computed as

$$P_i(\mathbf{s}) = \Phi\left[-\frac{\beta_i - \sum_{k=1}^{m} r_{ik}s_k}{\left(1 - \sum_{k=1}^{m} r_{ik}^2\right)^{0.5}}\right] \tag{22}$$

3.3 SRBDO/MSR. The proposed SRBDO/MSR (Fig. 1) adopts the same single-loop SRBDO approach in Eq. (14) except that P_{sys} is computed by the MSR method. It is thus formulated as

$$\min_{\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}},P_1^t,\ldots,P_n^t} \quad f(\mathbf{d},\boldsymbol{\mu}_{\mathbf{X}})$$

$$\text{s.t.} \quad g_i(\mathbf{d},\mathbf{X}(\mathbf{U}_i^t)) \geq 0, \quad i=1,\ldots,n \tag{23}$$

$$P_{\mathrm{sys}} = \begin{cases} \int_{\mathbf{s}} \mathbf{c}^{\mathrm{T}}\mathbf{p}(\mathbf{s}) f_{\mathbf{S}}(\mathbf{s}) d\mathbf{s} \le P_{\mathrm{sys}}^t & \text{dependent} \\ \mathbf{c}^{\mathrm{T}}\mathbf{p} \le P_{\mathrm{sys}}^t & \text{independent} \end{cases}$$

If the sensitivities of P_{sys} with respect to $\mathbf{d}$ and P_i^ts, $i=1,\ldots,n$ are available, one can use a gradient-based optimization algorithm for the SRBDO. As shown in Sec. 3.2, the MSR method provides the sensitivity of P_{sys} with respect to general parameters if the parameter sensitivities of component probabilities are available. For example, one can obtain such sensitivities using FORM [31]. Herein it is explained how the sensitivity with respect to P_is can be computed by the MSR method. First, the sensitivity of $P_i(\mathbf{s})$ with respect to the reliability index β_i is derived as

$$\frac{\partial P_i(\mathbf{s})}{\partial \beta_i} = -\frac{\varphi\left[-\left(\beta_i - \sum_{k=1}^{m} r_{ik}s_k\right)/\left(1-\sum_{k=1}^{m} r_{ik}^2\right)^{0.5}\right]}{\left(1-\sum_{k=1}^{m} r_{ik}^2\right)^{0.5}} \tag{24}$$

in which $\varphi(\cdot)$ denotes the PDF of the standard normal distribution. Then, the sensitivity with respect to the ith component probability is derived as

$$\frac{\partial P_i(\mathbf{s})}{\partial P_i} = \frac{\partial P_i(\mathbf{s})}{\partial \beta_i} \cdot \frac{\partial \beta_i}{\partial P_i} = -\frac{\partial P_i(\mathbf{s})}{\partial \beta_i} \cdot \frac{1}{\varphi(-\beta_i)} \tag{25}$$

This sensitivity is used for computing the sensitivity vector in Eq. (18) or Eq. (20).

4 Numerical Examples

In this section, three numerical examples are presented to demonstrate the capability and accuracy of the proposed SRBDO/MSR approach. In the first example, the optimal design of a combustion engine is obtained, in which the failure is described as a series system event. In the second example, the cross-sectional areas of the members of a statically indeterminate truss structure are determined for minimum total weight. A constraint is given on the probability of the system failure described by a cut-set system event. In the third example, the redistribution of the member forces caused by member failures is considered for the truss system in the second example. The results based on different RBDO approaches are compared for further investigations. Monte Carlo simulations are also performed to confirm the accuracy of the system failure probability computed by the MSR method.

4.1 Example 1: Design of an Internal Combustion Engine. This example adopted from Liang et al. [26] deals with the optimal design of the flat head of an internal combustion engine [32]. The objective is to find the mean values of the random design variables that maximize the "specific power" (or minimize the negative specific power). A constraint is given on the probability that the design will violate at least one of the requirements—a series system event. This SRBDO problem is formulated using the negative specific power as follows:

$$\min_{\boldsymbol{\mu}_{\mathbf{X}}} f(\boldsymbol{\mu}_{\mathbf{X}}) = -\frac{\mu_\omega}{120}[3688 \cdot \eta_t(\mu_{c_r},\mu_b,\mu_\omega) \cdot \eta_v(\mu_\omega,\mu_{d_I}) - \mathrm{FMEP}(\mu_{c_r},\mu_b,\mu_\omega)]$$

where

$$\mathrm{FMEP} = 4.826 \cdot (\mu_{c_r} - 9.2) + 7.97 + 0.253 \cdot [8V/(\pi N_c)]\mu_\omega(\mu_b)^{-2} + (9.7 \times 10^{-6}) \cdot \{[8V/(\pi N_c)]\mu_\omega(\mu_b)^{-2}\}^2$$

$$\eta_t = 0.8595 \cdot [1 - (\mu_{c_r})^{-0.33}] - S_v \cdot (1.5/\mu_\omega)^{0.5}$$

$$S_v = 0.83 \cdot [8 + 4\mu_{c_r} + 1.5 \cdot (\mu_{c_r} - 1)\mu_b^3 \pi N_c/V]/[(2+\mu_{c_r})\mu_b]$$

$$\eta_v = \eta_{vb} \cdot (1 + 5.96 \times 10^{-3}\mu_\omega^2)/\{1 + [(9.428 \times 10^{-5})4V/(\pi N_c C_s) \times (\mu_\omega/\mu_{d_I}^2)]^2\}$$

$$\eta_{vb} = \begin{cases} 1.067 - 0.038e^{(\mu_\omega - 5.25)} & \mu_\omega \ge 5.25 \\ 0.637 + 0.13\mu_\omega - 0.014\mu_\omega^2 + 0.00066\mu_\omega^3 & \mu_\omega < 5.25 \end{cases}$$

$$\text{s.t.} \quad P_{\mathrm{sys}} = P\left[\bigcup_{i=1}^{9} g_i(\mathbf{X}) \le 0\right] \le P_{\mathrm{sys}}^t$$

$$g_1 = 400 - 1.2N_c b \quad \text{(min. bore wall thickness)}$$

$$g_2 = b - [8V/(200\pi N_c)]^{0.5} \quad \text{(max. engine height)} \tag{26}$$

$$g_3 = 0.82b - d_I - d_E \quad \text{(valve geometry and structure)}$$

$$g_4 = d_E - 0.83d_I \quad \text{(min. valve diameter ratio)}$$

$$g_5 = 0.89d_I - d_E \quad \text{(max. valve diameter ratio)}$$

$$g_6 = 0.6C_s - (9.428 \times 10^{-5})(4V/\pi N_c)(\omega/d_I^2) \quad \text{(max. mech/index)}$$

$$g_7 = -0.045b - c_r + 13.2 \quad \text{(knock-limit compression ratio)}$$

$$g_8 = 6.5 - \omega \quad \text{(max. torque converter rpm)}$$

$$g_9 = 230.5Q\{0.8595 \cdot (1 - c_r^{-0.33}) - 0.83 \cdot [8 + 4c_r + 1.5 \times (c_r - 1)b^3\pi N_c/V]/[(2+c_r)b]\} - 3.6 \times 10^6 \quad \text{(max. fuel economy)}$$

where $V=1.859\times10^6$ mm^3, $Q=43{,}958$ kJ/kg, $C_s=0.44$, $N_c=4$, and $\mu_{(\cdot)}$ denotes the mean of the corresponding random variable in the subscript. The following five random variables are considered: the cylinder bore b, compression ratio c_r, exhaust valve diameter d_E, intake valve diameter d_I, and the revolution per minute at peak power (divided by 1000) denoted by ω. These are assumed to follow normal distributions. Table 1 shows the standard deviations of the random variables and the lower and upper bound values for their means, i.e., $\boldsymbol{\mu}_{\mathbf{X}}^L$ and $\boldsymbol{\mu}_{\mathbf{X}}^U$.

Liang et al. [26] first performed a PMA-based CRBDO (shown in Eq. (10)) for the given problem. For each of the nine requirements, the constraint on the component failure probability $P_i^t=0.00135$ (equivalent to target reliability index $\beta_i=3.0$) was assigned. The second column of Table 2 shows the optimal mean values and the corresponding maximum specific power 50.9713. The system failure probability was estimated as 0.006539 by Monte Carlo simulation (MCS) [26]. For the purpose of comparison, this MCS estimate was used as the constraint on P_{sys} during the single-loop SRBDO in Ref. [26] and SRBDO/MSR in this study. During SRBDO in Ref. [26], the "active set" strategy was introduced to deal with a convergence issue caused by small failure probabilities. They assigned "1" to active components whose failure probabilities P_i^t are greater than 10^{-7}, and "0" to the inactive components with smaller probabilities. The "inactive" components (those with "N/A" in Table 2) were excluded from the system failure probability calculations. The SRBDO/MSR in this

Table 1 Standard deviations of the random variables and bounds given on their means

Random variables	Std dev	Lower bounds	Upper bounds
Cylinder bore b (mm)	0.40	70	90
Intake valve diameter d_I (mm)	0.15	25	50
Exhaust valve diameter d_E (mm)	0.15	25	50
Compression ratio c_r	0.05	6	12
(rpm at peal power)/1000 ω	0.25	5	12

Table 2 Results of CRBDO [26], single-loop SRBDO [26], and SRBDO/MSR for combustion engine

		SRBDO		
	CRBDO by Liang et al. [26]	SRBDO by Liang et al. [26]	SRBDO/MSR	MCS for design by SRBDO/MSR
μ_b	82.1333	82.1419	82.1434	82.1434
μ_{d_I}	35.8430	35.8456	35.8394	35.8394
μ_{d_E}	30.3345	30.3641	30.3639	30.3639
μ_{c_r}	9.3446	9.3174	9.3194	9.3194
μ_ω	5.3141	5.3598	5.3621	5.3621
P_1^t	0.00135[a]	0.001448	0.001467	0.0014686
P_2^t	0.00135[a]	N/A	10^{-7}	0
P_3^t	0.00135[a]	0.001665	0.001558	0.0015627
P_4^t	0.00135[a]	0.000811	0.000778	0.0007713
P_5^t	0.00135[a]	N/A	10^{-7}	0
P_6^t	0.00135[a]	0.002370	0.002502	0.002503
P_7^t	0.00135[a]	0.000232	0.000266	0.0002573
P_8^t	0.00135[a]	N/A	0.000003	0.0000023
P_9^t	0.00135[a]	N/A	10^{-7}	0
P_{sys}^t	N/A	0.006539[a]	0.006539[a]	
P_{sys}	0.006539 (MCS)			0.006546
Max. power: $-f(\boldsymbol{\mu}_\mathbf{X})$	50.9713	51.1023	51.1014	51.1014

[a]Predetermined constraints.

study used a different optimizer [33] and did not experience the convergence issue, so the active set strategy was not used, but the lower bounds 10^{-7} were assigned on component probabilities P_i^t, $i=1,\ldots,9$ to facilitate the convergence. The component events whose probabilities are lower than the lower bound were not considered during the MSR analysis.

For the given problem, the optimal mean values and the maximum specific power by CRBDO are similar to those by SRBDOs. However, it should be noted that for a given SRBDO problem, the CRBDO approach may require repeated optimizations to find the level of constraints on the component failure probabilities that lead to the desired system-level reliability. It is also noted that the maximum specific power by CRBDO is smaller than those by SRBDOs even if the system failure probability is the same. This is because CRBDO approach (assigning fixed constraints on individual components) is generally more constrained than SRBDOs (assigning a constraint on system event, not on the individual components) at the same level of system reliability.

The comparison in Table 2 confirms that the two SRBDO approaches provide fairly close results for the series system problem. The small difference is caused by the upper bound approximation in SRBDO in Ref. [26]. According to the component failure probabilities of the optimal designs, the contribution of components 2, 5, 8, and 9 to system reliability is insignificant. The importance ranking of the other significant components is as follows: $6\rightarrow3\rightarrow1\rightarrow4\rightarrow7$. This ranking of component contributions is an important by-product of the SRBDO approaches. The fifth column of Table 2 shows the results of MCS (10^7 times; coefficient of variation (cov)=0.004) performed using the optimal design variables from SRBDO/MSR. The results confirm that the optimal design by SRBDO/MSR leads to the component/system failure probabilities that are compatible with the component failure probabilities found during optimization and with the assigned constraint on the system failure probability.

4.2 Example 2: SRBDO of an Indeterminate Truss Structure. The uniform applicability of SRBDO/MSR to general system problems is demonstrated by an SRBDO example of a statically indeterminate truss system by MacDonald and Mahadevan [34]. Figure 2 shows the geometry and the applied load of the truss system. The yielding failures of the six members are considered as component failure events. When the buckling failure modes, the dynamic effect of member damages, and the influence of the load redistribution during progressive failures [35] are neglected, the system fails when at least two members fail. The system failure event is described by the union of 15 minimal cut-sets: $\{C_k\}=\{(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$, each of which represents the joint failure of the corresponding members (see Fig. 2 for the member numbering choice).

In order to minimize the total weight of the structure, the objective function is defined such that it is proportional to the total weight of the members. The design variables are the cross-sectional areas of the members, A_i, $i=1,\ldots,6$, which are considered deterministic in this problem. The applied load F_A is assumed to follow a normal distribution with the mean of 4450 kN and a standard deviation of 445 kN, while the yield strengths of the members (in stress), F_i, $i=1,\ldots,6$, are assumed to be a normal distribution with the mean 745 MPa and the standard deviation 62 MPa. All random variables, $F_1,\ldots,F_6$ and F_A, are assumed to be statistically independent of each other. The member forces are derived in terms of the applied load assuming that the two diag-

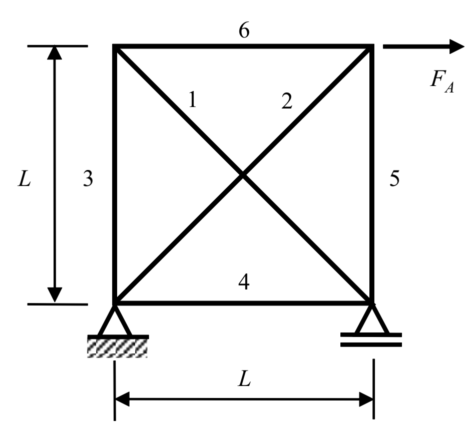

Fig. 2 A six-member indeterminate truss example

Table 3 Results of SRBDO [34] and SRBDO/MSR for the indeterminate truss system

Members	Area A_i ($\times 10^3$ mm^2)		Reliability index β_i	
	SRBDO by MacDonald and Mahadevan [34]	SRBDO/MSR	SRBDO by MacDonald and Mahadevan [34]	SRBDO/MSR
1	18.43	17.89	2.89	2.67
2	18.27	17.89	2.83	2.67
3	13.51	13.20	3.16	2.99
4	13.44	13.20	3.12	2.99
5	13.33	13.20	3.06	2.99
6	13.09	13.20	2.92	2.99

onal bars carry equal forces. The target system failure probability P_{sys}^t is given as 0.001. As a result, the SRBDO problem is formulated as

$$\min_{\mathbf{d}=\{A_1,\ldots,A_6\}} \quad f(\mathbf{d}) = \sqrt{2}(A_1 + A_2) + A_3 + A_4 + A_5 + A_6$$

$$\text{s.t.} \quad P_{\text{sys}} = P\left[\bigcup_{k=1}^{15} \bigcap_{i \in C_k} g_i(\mathbf{d},\mathbf{X}) \le 0\right] \le P_{\text{sys}}^t = 0.001$$

$$g_i(\mathbf{d},\mathbf{X}) = A_i F_i - 0.707 F_A, \quad i = 1,2 \tag{27}$$

$$A_i F_i - 0.500 F_A \quad i = 3, \ldots, 6$$

$$A_1, A_2, A_3, A_4, A_5, A_6 \ge 0$$

In the study by MacDonald and Mahadevan [34], a single-loop SRBDO approach shown in Eq. (14) was used except that the system failure probability was computed as follows. First, the probability of each cut-set was calculated as a parallel system using the product of conditional marginals method [36]. Considering the entire system event as a series system whose components are the cut-sets, the system failure probability was approximated by the first-order bounding formula in Eq. (12) with P_is replaced by the probabilities of the cut-sets.

The proposed SRBDO/MSR approach in Eq. (23) is applied to this example. The system failure probability and its sensitivities with respect to P_i^t are computed by the MSR method as explained in Sec. 3. The system failure probability is accurately estimated without using a bounding formula. The computed sensitivities facilitate the use of a gradient-based optimization algorithm. Table 3 compares the results by the two approaches. Except a slightly more conservative design in member 6, the SRBDO/MSR approach finds less conservative designs in all members while the same requirement on the system-level reliability is achieved. The minimum objective function value (i.e., minimum total weight) of the proposed approach is 103.36×10^3, which is less than that by the approximation method [34], 105.24×10^3. This is due to the overestimation of the system failure probability by the first-order bounding method, which results in a more conservative design than required. This is also evidenced by the lower reliability indexes of the component events by the proposed approach shown in Table 3. It is also noteworthy that due to the accurate system reliability estimates during the SRBDO/MSR, the symmetric conditions between diagonal members (1 and 2) and between nondiagonal members (3–6) give rise to symmetric results in the optimal design (i.e., cross-sectional areas) and the component failure probabilities (i.e., reliability indexes) as well. The system failure probability P_{sys} of the optimal cross-sectional areas found by SRBDO/MSR is evaluated as 0.001 by the MSR analysis and as 0.00107 by MCS (10^6 times, cov=0.03). Both estimates are fairly close to the given constraint 0.001.

According to the magnitude of component failure probabilities of the optimal design, the importance of the components is ranked in the order of $(1,2) \to (3,4,5,6)$. In order to quantify the relative importance of components based on their actual contributions to the system failure probability (not based on the magnitude of individual component events), the conditional probability importance measure (CIM) [23,37] of the ith component event

$$\text{CIM}_i = P(E_i | E_{\text{sys}}) = \frac{P(E_i E_{\text{sys}})}{P(E_{\text{sys}})} \tag{28}$$

can be used. This importance measure can be computed by the MSR method without significant additional computational cost. The system failure probability in the denominator is already available. Because the probability vector can be used once again, the only additional task required is to find the event vector for the new system event $E'_{\text{sys}} = E_i E_{\text{sys}}$. Figure 3 shows the CIMs of the truss members. The importance ranking is the same as that based on the individual component failure probabilities for this particular problem, but it should be noted that these rankings can be different in some cases. For example, if a constraint having high likelihood of violation does not contribute much to violating the system-level constraint, its CIM can be negligible despite its high failure probability.

Next, we assume all the random variables in the above example to follow the lognormal distributions with the same means and standard deviations. This is to investigate the effect of the types of the probabilistic distributions on the optimal design and to demonstrate the general applicability of the proposed method. The minimum objective function value is obtained as 105.46×10^3, which is slightly larger than that of the normal distribution case. Table 4 shows that the reliability indexes of the component events and the optimal cross-sectional areas of the lognormal distribution case are slightly larger than that of the normal distribution case. The system failure probability P_{sys} of the optimal design the SRBDO/MSR analysis is evaluated as 0.000998 by MCS (10^6 times, cov=0.032), which is close to the given constraint P_{sys}^t =0.001.

4.3 Example 3: SRBDO of an Indeterminate Truss Structure Considering Progressive Failure.

In this example, the SRBDO problem in Example 2 is re-investigated with consideration of load redistribution in the truss system caused by member failures. This load redistribution can cause a progressive failure of

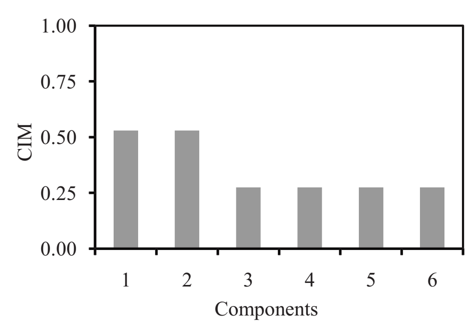

Fig. 3 Conditional probability importance measures of the truss members

Table 4 Results of SRBDO/MSR for normal and lognormal distribution cases

Members	Area A_i ($\times 10^3$ mm^2)		Reliability index β_i	
	Normal	Lognormal	Normal	Lognormal
1	17.89	18.18	2.67	2.79
2	17.89	18.18	2.67	2.79
3	13.20	13.51	2.99	3.17
4	13.20	13.51	2.99	3.17
5	13.20	13.51	2.99	3.17
6	13.20	13.51	2.99	3.17

the system. All the parameters are the same as Example 2. The complexity of estimating the likelihood of this system event arises from the fact that the failures of the remaining members should be described as new component events due to the load redistribution. Figure 4 shows the numbering choice of the component failure events defined for the members in the original structure and the structures with one failed member.

The structure survives if (1) no member fails in the original configuration or (2) one member fails but no further member failures take place. Using the component numbering choice shown in Fig. 4, the probability of system survival $\bar{E}_{\text{sys}}$ is described as

$$\begin{aligned} P(\bar{E}_{\text{sys}}) = P[&\bar{E}_1\bar{E}_2\bar{E}_3\bar{E}_4\bar{E}_5\bar{E}_6 \\ &\cup (E_1\bar{E}_2\bar{E}_3\bar{E}_4\bar{E}_5\bar{E}_6)(\bar{E}_7\bar{E}_8\bar{E}_9\bar{E}_{10}\bar{E}_{11}) \\ &\cup (\bar{E}_1E_2\bar{E}_3\bar{E}_4\bar{E}_5\bar{E}_6)(\bar{E}_{12}\bar{E}_{13}\bar{E}_{14}\bar{E}_{15}\bar{E}_{16}) \\ &\cup (\bar{E}_1\bar{E}_2E_3\bar{E}_4\bar{E}_5\bar{E}_6)(\bar{E}_{17}\bar{E}_{18}\bar{E}_{19}\bar{E}_{20}\bar{E}_{21}) \\ &\cup (\bar{E}_1\bar{E}_2\bar{E}_3E_4\bar{E}_5\bar{E}_6)(\bar{E}_{22}\bar{E}_{23}\bar{E}_{24}\bar{E}_{25}\bar{E}_{26}) \\ &\cup (\bar{E}_1\bar{E}_2\bar{E}_3\bar{E}_4E_5\bar{E}_6)(\bar{E}_{27}\bar{E}_{28}\bar{E}_{29}\bar{E}_{30}\bar{E}_{31}) \\ &\cup (\bar{E}_1\bar{E}_2\bar{E}_3\bar{E}_4\bar{E}_5E_6)(\bar{E}_{32}\bar{E}_{33}\bar{E}_{34}\bar{E}_{35}\bar{E}_{36})] \end{aligned} \tag{29}$$

in which E_i and $\bar{E}_i$, respectively, denote the failure and survival event of the ith component. This is a link-set system event consisting of 36 components. The size of $\mathbf{c}$ and $\mathbf{p}$ is $2^{36} \cong 6.87 \times 10^{10}$. However, the size of the vectors used in MSR analysis can be further reduced as follows. Due to the mutual exclusiveness of the seven link-sets, the probability can be computed as the sum of the probabilities of the individual link-sets, which reduces the maximum number of components appearing in an MSR analysis from 36 to 11. It can be further reduced by considering the fact that some link-sets include component events defined for the same member. For example, the component events $\bar{E}_1$ and $\bar{E}_{12}$ are defined for the same member, as shown in Fig. 4. Since their limit-state functions indicate $\bar{E}_{12} \subset \bar{E}_1$ for positive values of F_1 and F_A, $\bar{E}_1\bar{E}_{12}$ is simplified to $\bar{E}_{12}$. As a result, the system reliability can be computed as

$$\begin{aligned} P(\bar{E}_{\text{sys}}) = {} & P(\bar{E}_1\bar{E}_2\bar{E}_3\bar{E}_4\bar{E}_5\bar{E}_6) + P(E_1\bar{E}_3\bar{E}_4\bar{E}_6\bar{E}_7\bar{E}_{10}) \\ & + P(E_2\bar{E}_5\bar{E}_{12}\bar{E}_{13}\bar{E}_{14}\bar{E}_{16}) + P(\bar{E}_1E_3\bar{E}_4\bar{E}_6\bar{E}_{18}\bar{E}_{20}) \\ & + P(\bar{E}_1\bar{E}_3E_4\bar{E}_6\bar{E}_{23}\bar{E}_{25}) + P(\bar{E}_2E_5\bar{E}_{27}\bar{E}_{29}\bar{E}_{30}\bar{E}_{31}) \\ & + P(\bar{E}_1\bar{E}_3\bar{E}_4E_6\bar{E}_{33}\bar{E}_{36}) \end{aligned} \tag{30}$$

This system decomposition reduces the maximum number of components appearing an MSR analysis to 6. The size of the vectors is only $2^6 = 64$.

Noting some component events having the same limit-state functions, Eq. (30) is rewritten using 12 distinct component events as follows:

$$\begin{aligned} P(\bar{E}_{\text{sys}}) = {} & P(\bar{E}_1\bar{E}_2\bar{E}_3\bar{E}_4\bar{E}_5\bar{E}_6) + P(E_1\bar{E}_3\bar{E}_4\bar{E}_6\bar{E}_7\bar{E}_{10}) \\ & + P(E_2\bar{E}_5\bar{E}_{12}\bar{E}_{13}\bar{E}_{14}\bar{E}_{16}) + P(\bar{E}_1E_3\bar{E}_4\bar{E}_6\bar{E}_7\bar{E}_{10}) \\ & + P(\bar{E}_1\bar{E}_3E_4\bar{E}_6\bar{E}_7\bar{E}_{10}) + P(\bar{E}_2E_5\bar{E}_{12}\bar{E}_{13}\bar{E}_{14}\bar{E}_{16}) \\ & + P(\bar{E}_1\bar{E}_3\bar{E}_4E_6\bar{E}_7\bar{E}_{10}) \end{aligned} \tag{31}$$

Thus, the SRBDO problem is formulated as

$$\min_{\mathbf{d}=\{A_1,\ldots,A_6\}} f(\mathbf{d}) = \sqrt{2}(A_1 + A_2) + A_3 + A_4 + A_5 + A_6$$

$$\text{s.t.} \quad P_{\text{sys}} = 1 - P(\bar{E}_{\text{sys}}) = 1 - \sum_{k=1}^{7} P\Big(\bigcap_{i \in L_k} g_i(\mathbf{d},\mathbf{X}) > 0\Big) \le 0.001$$

$$\begin{aligned} g_i(\mathbf{d},\mathbf{X}) = {} & A_iF_i - 0.707F_A, \quad i = 1,2 \\ & A_iF_i - 0.500F_A, \quad i = 3,\ldots,6 \\ & A_2F_2 - 1.414F_A, \quad i = 7 \\ & A_5F_5 - 1.000F_A, \quad i = 10 \\ & A_1F_1 - 1.414F_A, \quad i = 12 \\ & A_3F_3 - 1.000F_A, \quad i = 13 \\ & A_4F_4 - 1.000F_A, \quad i = 14 \\ & A_6F_6 - 1.000F_A, \quad i = 16 \\ & A_1,A_2,A_3,A_4,A_5,A_6 \ge 0 \end{aligned} \tag{32}$$

where L_k, $k = 1,\ldots,7$ is the component index set from Eq. (31), that is, $\{L_k\} = \{(1,2,3,4,5,6),(1,3,4,6,7,10),(2,5,12,13,14,16),(1,3,4,6,7,10),(1,3,4,6,7,10),(2,5,12,13,14,16),(1,3,4,6,7,10)\}$.

Table 5 compares the results of the SRBDO in Eq. (32) (denoted by "Ex. 3" in the following tables) with those given in Example 2 (Ex. 2). It is seen that the optimal cross-sectional areas increase significantly as the effect of load redistribution is considered. The objective function value also increases from 103.36×10^3 to 114.13×10^3. This implies that neglecting the load redistribution during an SRBDO may result in a design that does not satisfy the system-level safety criteria. As expected, when the load redistribution is considered, the system failure probability of the optimal design from Example 2 is estimated as 0.01208 by MSR

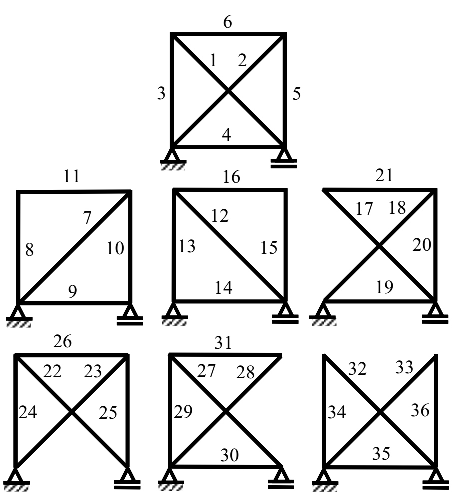

Fig. 4 Component failure events defined for the original system and systems with failed members

Table 5 Results of SRBDO/MSR of the truss system with/without consideration of load redistribution

Member (Fig. 2)	Component events (Eq. (31))	Area A_i ($\times 10^3$ mm^2)		Reliability index β_i	
		Ex. 2	Ex. 3	Ex. 2	Ex. 3[a]
1	1, 12	17.89	19.94	2.668	3.48, −2.07
2	2, 7	17.89	19.94	2.668	3.48, −1.78
3	3, 13	13.20	14.44	2.987	3.65, −1.95
4	4, 14	13.20	14.44	2.987	3.65, −1.95
5	5, 10	13.20	14.44	2.987	3.65, −1.62
6	6, 16	13.20	14.44	2.987	3.65, −1.95

[a]Two reliability indexes correspond to the component events in the second column (in the same order).

Table 6 Optimal design and correlation between member yield strengths

		Area A_i ($\times 10^3$ mm^2)							
		ρ=0.00[a]		ρ=0.25		ρ=0.50		ρ=0.75	
		Ex. 2	Ex. 3	Ex. 2	Ex. 3	Ex. 2	Ex. 3	Ex. 2	Ex. 3
Members (Fig. 2)	1	17.89	19.94	18.03	19.90	18.14	19.83	18.31	19.68
	2	17.89	19.94	18.03	19.90	18.14	19.83	18.31	19.68
	3	13.20	14.44	13.53	14.40	13.91	14.33	14.28	14.16
	4	13.20	14.44	13.53	14.40	13.91	14.33	14.28	14.16
	5	13.20	14.44	13.53	14.40	13.91	14.33	14.28	14.16
	6	13.20	14.44	13.53	14.40	13.91	14.33	14.28	14.16
Objective function		103.36	114.13	105.12	113.93	106.93	113.42	108.90	112.34

[a]Results in Table 5.

analysis and 0.01205 by MCS (10^6 times; cov=0.009), which clearly exceed the given constraint 0.001. By contrast, the system failure probability of the optimal design in this example is estimated as 0.001 by MSR and 0.000986 by MCS (10^6 times, cov =0.0318), which are close to the given constraint.

The impact of statistical correlation between random yielding strengths F_is on the optimal design is investigated by varying their correlation coefficients $\rho_{i,j}$. For simplicity, the correlation coefficients are assumed to be uniform, i.e., $\rho_{i,j}=\rho$. The SRBDO problems in Eqs. (27) and (32) are solved again with correlation coefficient ρ varying from 0.00 to 0.75. The optimal cross-sectional areas and the objective function values are shown in Table 6. It is seen that considering the effect of load redistribution results in more conservative designs for all levels of correlation considered. It is also observed that the higher correlation among member yield strengths increases the cross-sectional areas when redistribution is not considered but decreases if redistribution considered. Figure 5 presents this trend more clearly by showing the objective function values of the SRBDOs.

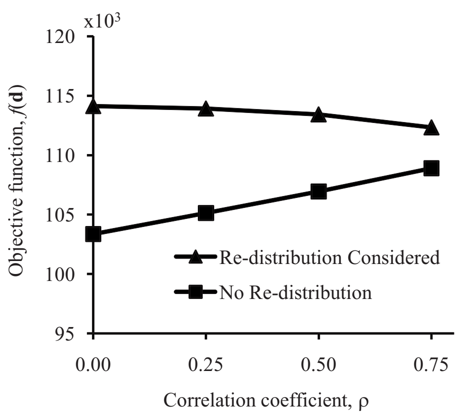

Fig. 5 Objective functions versus correlation between member yield strengths

5 Summary and Conclusions

In this study, an efficient and accurate system reliability-based design optimization approach is developed by integrating a single-loop RBDO algorithm with the recently developed matrix-based system reliability method. The use of the MSR method improves the efficiency and accuracy in computing the system probability and its sensitivities in the existing single-loop SRBDO approach [26]. The MSR method enables us to compute the probabilities of general system events including series, parallel, cut-set, and link-set systems in a uniform manner without using approximate bounds or random samplings. It can account for statistical dependence between component events and can compute the sensitivities of the system failure probability with respect to various parameters as well, which facilitates the use of gradient-based optimization algorithms. Three numerical examples demonstrate the uniform applicability of the proposed SRBDO/MSR approach to series, cut-set, and link-set systems. It is seen that the accuracy of system reliability analysis by the MSR method enables us to obtain less conservative optimal design than SRBDO algorithms using upper bounds. The effect of load redistribution by member failures on the optimal designs is investigated as well. In each example, the accuracy of the MSR method is verified by Monte Carlo simulations. It is noteworthy that the MSR in this study employs FORM for component reliability analysis and MSR accurately estimates *system probability* based on the provided component probabilities. If the component reliability analysis FORM provides inaccurate component probabilities due to the nonlinearity of the limit-state functions, this inaccuracy will affect the accuracy of any SRBDO algorithms including SRBDO/MSR. Potential topics for future studies include improving the accuracy of SRBDOs using the second-order reliability method, handling a mixed set of continuous and discrete random variables in SRBDOs, and time variant SRBDOs [38,39].

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Nomenclature

$\mathbf{c}$ = event vector
$\mathbf{d}$ = deterministic design variables
$f(\cdot)$ = objective function
$g_i(\mathbf{d},\mathbf{X})$ = limit-state (or performance) function of the ith failure mode
$\mathbf{p}$ = probability vector
P_i = actual failure probability of the ith mode
P_i^t = target failure probability of the ith mode
P_{sys} = actual system failure probability
P_{sys}^t = target system failure probability
$\mathbf{S}$ = common source random variables
$\mathbf{U}_i^*$ = most probable failure point of the ith mode
$\mathbf{X}$ = random variables
$\hat{\boldsymbol{\alpha}}_i$ = negative normalized gradient vector
β_i = reliability index
β_i^t = target reliability index
$\boldsymbol{\mu}_{\mathbf{X}}$ = vector of means of $\mathbf{X}$

Appendix: Illustrative Example of MSR Method

Consider three system events, each of which consists of five component events, E_i, $i=1,\ldots,5$

$$E_{\text{sys}} = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \quad \text{(series)}$$

$$E_{\text{sys}} = E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \quad \text{(parallel)} \tag{A1}$$

$$E_{\text{sys}} = (E_1 \cup E_2 \cup E_3) \cap (E_2 \cup E_3 \cup E_4) \cap (E_3 \cup E_4 \cup E_5) \quad \text{(link-set)}$$

In this illustrative example, the probabilities of the system events, $P(E_{\text{sys}})$, are computed by the MSR method based on the results of the component reliability analyses by first-order reliability method. After FORM analysis, each component event is approximately described by

$$E_i: \quad Z_i \le -\beta_i, \quad i=1,\ldots,5 \tag{A2}$$

where Z_i is correlated standard normal random variable; and β_i is the FORM reliability index of E_i, $i=1,\ldots,5$. The correlation coefficient between Z_i and Z_j, $i \neq j$ are computed by the inner-product of the negative normalized gradient vectors at the corresponding MPPs [6]. In this example, suppose $\beta_i=3$, $i=1,\ldots,5$ and the inner products give the correlation coefficient matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 0.89 & 0.88 & 0.87 & 0.86 \\ 0.89 & 1 & 0.90 & 0.90 & 0.90 \\ 0.88 & 0.99 & 1 & 0.90 & 0.90 \\ 0.87 & 0.90 & 0.90 & 1 & 0.90 \\ 0.86 & 0.90 & 0.90 & 0.90 & 1 \end{bmatrix} \tag{A3}$$

The MSR method computes the probabilities of the system events in Eq. (A1) by the matrix formulation in Eq. (16). The numerical integration requires three tasks: (a) describing $\mathbf{R}$ approximately by use of a generalized DS model and identifying common source random variables $\mathbf{S}$, (b) constructing the event vector $\mathbf{c}$, and (c) computing the conditional probability vector $\mathbf{p}(\mathbf{s})$.

First, the correlation coefficients in $\mathbf{R}$ are fitted by those constructed by a generalized DS model, i.e., $\rho_{ij}=\Sigma_{k=1}^{m}(r_{ik}.r_{jk})$ with the minimum error. When one CSRV is used, the coefficients in the generalized DS model in Eq. (17) are

$$r_{11}=0.9223, \quad r_{21}=0.9539, \quad r_{31}=0.9541, \quad r_{41}=0.9432, \quad r_{51}=0.9435 \tag{A4}$$

When two CSRVs are used for improved accuracy, the coefficients are obtained as

$$\begin{aligned} &r_{11}=0.9262, \quad r_{21}=0.6989, \quad r_{31}=0.6801, \quad r_{41}=0.6632, \quad r_{51}=0.6427 \\ &r_{12}=0.3769, \quad r_{22}=0.6436, \quad r_{32}=0.6614, \quad r_{42}=0.6782, \quad r_{52}=0.6998 \end{aligned} \tag{A5}$$

The joint PDF of CSRVs in Eq. (16), i.e., $f_{\mathbf{S}}(\mathbf{s})$ is $\varphi(s_1)$ and $\varphi(s_1)\varphi(s_2)$ for one- and two-CSRV cases, respectively, in which $\varphi(\cdot)$ denotes the PDF of the standard normal distribution.

Second, the event vector $\mathbf{c}$ is constructed for each of the system events. The event vectors for the five component events are first constructed by the sequential matrix-based procedures proposed in Ref. [35]

$$\mathbf{C}_{[1]} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C}_{[i]} = \begin{bmatrix} \mathbf{C}_{[i-1]} & \mathbf{1} \\ \mathbf{C}_{[i-1]} & \mathbf{0} \end{bmatrix} \quad \text{for} \quad i=1,2,\ldots,5 \tag{A6}$$

where $\mathbf{0}$ and $\mathbf{1}$ denote the column vectors of 2^{i-1} zeros and ones, respectively. When the iterative procedure is completed, the ith column of $\mathbf{C}_{[5]}$ is the event vector of the ith component event E_i, $i=1,\ldots,5$. As a result, the event vectors of the five components are obtained as

$$\begin{aligned} \mathbf{c}^{E_1} &= [10101010101010101010101010101010]^{\mathrm{T}} \\ \mathbf{c}^{E_2} &= [11001100110011001100110011001100]^{\mathrm{T}} \\ \mathbf{c}^{E_3} &= [11110000111100001111000011110000]^{\mathrm{T}} \\ \mathbf{c}^{E_4} &= [11111111000000001111111100000000]^{\mathrm{T}} \\ \mathbf{c}^{E_5} &= [11111111111111110000000000000000]^{\mathrm{T}} \end{aligned} \tag{A7}$$

Then, the event vector of the system event E_{sys} is obtained by matrix-based procedures employing the event vectors of the components. For example, the event vector for the complementary event of E, the intersection and the union of the component events are obtained as follows:

$$\begin{aligned} \mathbf{c}^{\bar{E}} &= \mathbf{1} - \mathbf{c}^{E} \\ \mathbf{c}^{E_1 \cdots E_n} &= \mathbf{c}^{E_1} .* \mathbf{c}^{E_2} .* \cdots .* \mathbf{c}^{E_n} \\ \mathbf{c}^{E_1 \cup \cdots \cup E_n} &= \mathbf{1} - (\mathbf{1} - \mathbf{c}^{E_1}) .* (\mathbf{1} - \mathbf{c}^{E_2}) .* \cdots .* (\mathbf{1} - \mathbf{c}^{E_n}) \end{aligned} \tag{A8}$$

where $.*$ denotes the element-wise multiplication of two vectors.

Finally, the conditional probability vector $\mathbf{p}(\mathbf{s})$ is constructed by the following matrix-based procedure:

$$\begin{aligned} \mathbf{p}_{[1]}(\mathbf{s}) &= [P_1(\mathbf{s}) 1 - P_1(\mathbf{s})]^{\mathrm{T}} \\ \mathbf{p}_{[i]}(\mathbf{s}) &= \begin{bmatrix} \mathbf{p}_{[i-1]}(\mathbf{s}) \cdot P_i(\mathbf{s}) \\ \mathbf{p}_{[i-1]}(\mathbf{s}) \cdot [1 - P_i(\mathbf{s})] \end{bmatrix} \quad \text{for} \quad i=2,\ldots,5 \end{aligned} \tag{A9}$$

where $P_i(\mathbf{s})$ is the conditional probability of the ith component given $\mathbf{S}=\mathbf{s}$, which is computed by Eq. (22) employing the reliability indexes $\beta_i=3$ and the generalized DS model coefficients in Eq. (A4) or Eq. (A5). When the sequential matrix-based procedure in

Table 7 System probabilities computed by MSR, MCS, and bounding formula ($\times 10^{-3}$)

System events	Bi-component bounds		MSR: No. of CSRVs		MCS ($N=10^7$ times)	
	Lower bound	Upper bound	1	2	$P(E_{sys})$	cov
Series	2.309	4.338	3.528	3.526	3.532	0.005
Parallel	N/A	N/A	0.2314	0.2318	0.2329	0.021
Link-set	N/A	N/A	1.738	1.739	1.764	0.008

Eq. (A9) is completed, $\mathbf{p}_{[5]}(\mathbf{s})$ is used as $\mathbf{p}(\mathbf{s})$ in Eq. (16).

Table 7 shows the results of the system reliability analysis by the MSR method, Monte Carlo simulations and the bi-component bounding formula [27]. Close agreements between the results by MSR method and those by MCS confirm the accuracy of the MSR method for the given example while the bi-component bounds show significant width for the series system.

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