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International Journal of Heat and Mass Transfer 51 (2008) 3018–3024
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Heat flux field for one spherical inhomogeneity embedded in a functionally graded material matrix

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Received 19 January 2007; received in revised form 10 September 2007
Available online 19 November 2007

Abstract

The heat flux field for a single particle embedded in a graded material is derived by using the equivalent inclusion method. A linearly distributed prescribed heat flux field is introduced to represent the material mismatch between the particle and the surrounding graded materials. By using Green's function technique, an explicit solution is obtained for the heat flux field in both the particle and the graded material. Comparison of the present solution with finite element results illustrates the accuracy and limitation of this solution.
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Keywords: Functionally graded materials; Equivalent inclusion method; Inhomogeneity; Thermal conduction; Heat transfer

1. Introduction

Functionally graded materials (FGMs) have attracted significant attention amongst researchers and engineers due to their unique thermomechanical properties and microstructural design [1–6]. They have been manufactured into thermal barrier coatings, tribological coatings, and energy conversion materials. Although graded materials microscopically exhibit a heterogeneous microstructure, macroscopically their effective material properties continuously vary in the gradation direction but keep constant in the plane normal to the gradation direction. Material property variations can also be found in some civil engineering materials and constructed facilities. For instance, asphalt pavements exhibit a severe age hardening gradient near the surface due to the environmental factors such as oxidative hardening. In addition, above the groundwater table, the moisture content for subgrade materials also varies in the vertical direction, which induces a material property gradient.

Because in most applications FGMs are subjected to thermal loading, heat conduction has been widely investigated by various numerical methods, such as the finite element method [7], boundary element method [8], and meshless method [9]. However, because FGMs generally have a complex microstructure but the accuracy of numerical simulations depends on the quality of discretization aspects, it is not straightforward to extend these results to general cases. Thus, analytical methods become a very valuable tool for model verification, and ultimately to gain a better insight into heat conduction in FGMs.

The inhomogeneity problem involving a single particle embedded in a homogeneous matrix is a fundamental problem for theorists in a variety of fields: materials science, solid-state physics, and mechanics of composites. Eshelby's formulation [10,11] for an ellipsoidal inclusion

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doi:10.1016/j.ijheatmasstransfer.2007.09.027

embedded in an unbounded matrix with a uniform, far-field loading provided a cornerstone for micromechanics. Since then, mechanical solutions for the inhomogeneity problem have received considerable attention. For instance, Willis [12] developed an elastic solution for a single inclusion embedded in an anisotropic matrix; Gilormini and Montheillet [13] considered the strain rates and stresses for an inclusion in a viscous matrix; and Rodin [14], Kawashita and Nozaki [15], and Zheng et al. [16] studied the effect of the shape of the inhomogeneity on the stress and strain distribution. Furthermore, Hatta and Taya [17] extended Eshelby's method to heat conduction problems. However, all of these works focus on an inhomogeneity embedded in a homogeneous matrix.

This work investigates the inhomogeneity problem for one spherical particle embedded in a graded material under a uniform heat flux in the far field. Herein, the size of the particle considered is assumed to be much smaller than the size of the graded matrix. Because the disturbed heat flux field due to the inhomogeneity is localized in the neighborhood of the particle, and the material property variation is continuous and differentiable in the gradation direction, we use a parabolic variation of thermal conductivity to approximate the material gradation of the matrix in the neighborhood of the particle. Then Eshelby's equivalent inclusion method is used to solve for the disturbed heat flux field, in which the effect of material mismatch between the particle and the graded matrix on the heat flux field is simulated by introducing a prescribed heat flux. The heat flux distribution from the present analytic solution is in excellent agreement with finite element results.

The remainder of this paper is organized as follows. Section 2 derives the heat flux field in an FGM due to a prescribed heat flux field in a spherical domain. Section 3 briefly reviews the equivalent inclusion method and formulates the heat flux field for a particle embedded in an unbounded graded material. Section 4 presents a verification of the present solution to finite element results. Finally, concluding remarks are given in Section 5.

2. Heat flux field in an FGM due to a prescribed heat flux

In an FGM, when a prescribed distributed heat flux (thermal doublet) denoted as $q^*(\mathbf{x})$ is applied in a spherical domain Ω along the gradation direction, i.e. x_3 direction (see Fig. 1), a local heat flux field will be induced in the neighborhood of the particle domain. The relationship between the heat flux and the temperature gradient is given by the Fourier law as

$$q_i(\mathbf{x}) = q^*(\mathbf{x})\delta_{i3} - k(\mathbf{x})T_{,i}(\mathbf{x}), \tag{1}$$

where $q^*(\mathbf{x}) = 0$ for $\mathbf{x} \in D - \Omega$. Here D denotes the total FGM domain. At steady state without heat generation, the heat flux field satisfies the following differential equation of heat conduction:

$$q_{i,i}(\mathbf{x}) = 0. \tag{2}$$

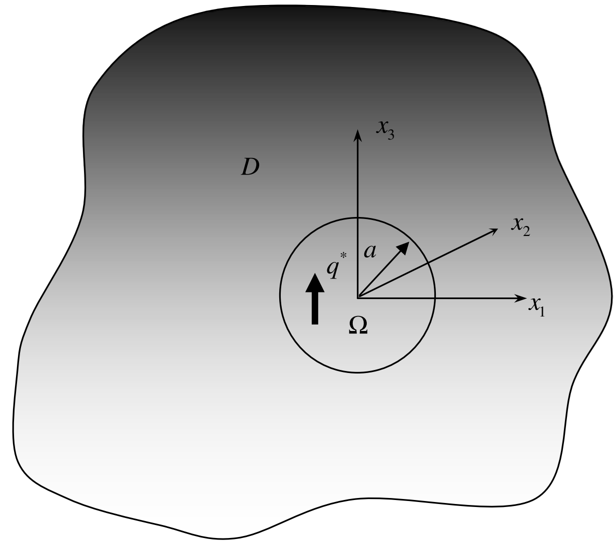

Fig. 1. An FGM subjected to a prescribed heat flux in a spherical domain.

The variation of the FGM properties is assumed to be continuous and differentiable in the gradation direction, so that the thermal conductivity distribution can be written as

$$k(x_3) = k^0 + k'(0)x_3 + \frac{1}{2}k''(0)x_3^2 + \cdots, \tag{3}$$

where k^0 represents the thermal conductivity at the origin, and the primes in k' and k'' denote the first and second derivatives of k, respectively. Because the material properties in far field only produce a minor effect on the heat flux field in the neighborhood of the particle, the lower order terms of (3) have dominant effects on the solution, and thus (3) is rewritten as

$$k(x_3) = k^0(1 + \alpha x_3)^2 + \mathrm{O}(x_3^2), \tag{4}$$

where the material variation parameter $\alpha = 0.5k'(0)/k^0$, and the higher order terms $\mathrm{O}(x_3^2)$ will be disregarded for the convenience of derivation. It is noted that accuracy of approximation in (4) also depends on the magnitude of the material gradient. In this paper, we assume

$$\alpha a << 1. \tag{5}$$

Substituting (4) into (1), and (1) into (2), one obtains

$$k^0[(1 + \alpha x_3)^2 T_{,i}]_{,i} = \frac{\partial q^*}{\partial x_3}, \tag{6}$$

where the higher order terms $\mathrm{O}(x_3^2)$ in (4) are disregarded. By introducing a new variable, namely

$$U = (1 + \alpha x_3)T. \tag{7}$$

Eq. (6) can be rewritten as

$$\nabla^2 U = \frac{1}{k^0(1 + \alpha x_3)} \frac{\partial q^*}{\partial x_3}. \tag{8}$$

Because temperature and heat flux fields induced by the prescribed heat flux decay to zero rapidly in the far field, by using Green's function technique, the variable U can be expressed in an integral form:

$$U(\mathbf{x}) = -\int_D \frac{G(\mathbf{x}-\mathbf{x}')}{k^0(1+\alpha x_3')} \frac{\partial q^*}{\partial x_3'} d\mathbf{x}', \tag{9}$$

where Green's function $G(\mathbf{x}-\mathbf{x}')$ describes the response at point $\mathbf{x}$ due to the source at point $\mathbf{x}'$ in the infinite domain, which is written as

$$G(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}. \tag{10}$$

Because $q^* = 0$ for $|\mathbf{x}'| \to \infty$, using Green's theorem in (9), one obtains

$$U(\mathbf{x}) = -\int_D \frac{\partial G(\mathbf{x}-\mathbf{x}')}{\partial x_3} \frac{q^*}{k^0(1+\alpha x_3')} d\mathbf{x}' - \int_D G(\mathbf{x}-\mathbf{x}') \frac{\alpha q^*}{k^0(1+\alpha x_3')^2} d\mathbf{x}', \tag{11}$$

where $\partial G(\mathbf{x}-\mathbf{x}')/\partial x_3 = -\partial G(\mathbf{x}-\mathbf{x}')/\partial x_3'$ is used. Combining (7) and (1) with (4) provides

$$q_i(\mathbf{x}) = q^*(\mathbf{x})\delta_{i3} - k^0(1+\alpha x_3)U_{,i}(\mathbf{x}) + k^0\alpha\delta_{i3}U(\mathbf{x}). \tag{12}$$

Next, it is assumed that the prescribed heat flux is a linear function of x_3 in the domain Ω as

$$q^*(\mathbf{x}) = q^0 + \tilde{q}x_3 \quad \text{for } \mathbf{x} \in \Omega, \tag{13}$$

where $\tilde{q}$ represents the linear coefficient of the prescribed heat flux distribution. Substitution of (13) into (11) renders

$$U(\mathbf{x}) = -\int_D \frac{1}{k^0} \frac{\partial G(\mathbf{x}-\mathbf{x}')}{\partial x_3} \left[q^0 + (\tilde{q} - \alpha q^0)x_3' + \mathrm{O}(x_3'^2)\right] d\mathbf{x}' - \int_D \frac{\alpha}{k^0} G(\mathbf{x}-\mathbf{x}') \times \left[q^0 + (\tilde{q} - 2\alpha q^0)x_3' + \mathrm{O}(x_3'^2)\right] d\mathbf{x}'. \tag{14}$$

After disregarding the higher order terms $\mathrm{O}(x_3^2)$ in (14), $U(\mathbf{x})$ can be explicitly integrated as

$$U(\mathbf{x}) = \begin{cases} \frac{1}{15k^0}\left[\begin{array}{l} 5\rho a q^0(\rho n_3 - 5\alpha a) \\ -\rho^3 a^2(1-3n_3^2)(\tilde{q}-\alpha q^0) \\ -\alpha\rho^2 a^3 n_3(\tilde{q}-2\alpha q^0) \end{array}\right], & \mathbf{x} \notin \Omega, \\ \frac{1}{30k^0}\left[\begin{array}{l} q^0\left(10x_3 - 5\alpha\left(3a^2 - |\mathbf{x}|^2\right)\right) \\ -(\tilde{q}-\alpha q^0)(5a^2 - 3|\mathbf{x}|^2 - 6x_3^2) \\ -\alpha(\tilde{q}-2\alpha q^0)(5a^2 - 3|\mathbf{x}|^2)x_3 \end{array}\right], & \mathbf{x} \in \Omega, \end{cases} \tag{15}$$

where $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$, and $\rho = a/|\mathbf{x}|$. Similarly, one can further obtain the gradient of the above function as

$$U_{,i}(\mathbf{x}) = \begin{cases} \frac{1}{15k^0}\left[\begin{array}{l} 5q^0\rho^2(\rho(\delta_{i3} - 3n_3n_i) + \alpha a n_i) \\ +3(\tilde{q}-\alpha q^0)\rho^4 a \\ \times(2\delta_{i3}n_3 + n_i - 5n_3^2 n_i) \\ -\alpha(\tilde{q}-2\alpha q^0)\rho^3 a^2(\delta_{i3} - 3n_3n_i) \end{array}\right], & \mathbf{x} \notin \Omega, \\ \frac{1}{30k^0}\left(\begin{array}{l} 10q^0(\delta_{i3} + \alpha x_i) + 6(\tilde{q}-\alpha q^0) \\ \times(2\delta_{i3}x_3 + x_i) - \alpha(\tilde{q}-2\alpha q^0) \\ \times\left[\left(5a^2 - 3|\mathbf{x}|^2\right)\delta_{i3} - 6x_3x_i\right] \end{array}\right), & \mathbf{x} \in \Omega. \end{cases} \tag{16}$$

Substituting (15) and (16) into (12), one obtains the explicit form of the heat flux field.

3. Single inhomogeneity in a functionally graded material

Consider an unbounded FGM domain with heat conductivity, $k(x_3)$, containing a single spherical inhomogeneity Ω (see Fig. 2) with heat conductivity k^1, radius a, with its center located at the origin. A uniform heat flux field q^∞ is applied in the x_3 direction in the far field.

Because the FGM is homogeneous in the $x_1 - x_2$ plane, if the particle did not exist, then the heat flux field would be uniform. However, a disturbance in the heat flux field q_i' will be induced by the presence of the particle. Then the local heat flux field can be denoted by two parts:

$$q_i(\mathbf{x}) = q^\infty\delta_{i3} + q_i'(\mathbf{x}). \tag{17}$$

The constitutive relation is written as

$$q^\infty\delta_{i3} + q_i'(\mathbf{x}) = -k(x_3)T_{,i}(\mathbf{x}), \quad \mathbf{x} \notin \Omega, \tag{18}$$

$$q^\infty\delta_{i3} + q_i'(\mathbf{x}) = -k^1T_{,i}(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{19}$$

The equivalent inclusion method is employed to derive the heat flux disturbance using a prescribed heat flux q^* in the particle domain Ω to simulate the material mismatch, so that the constitutive relation in the particle domain becomes

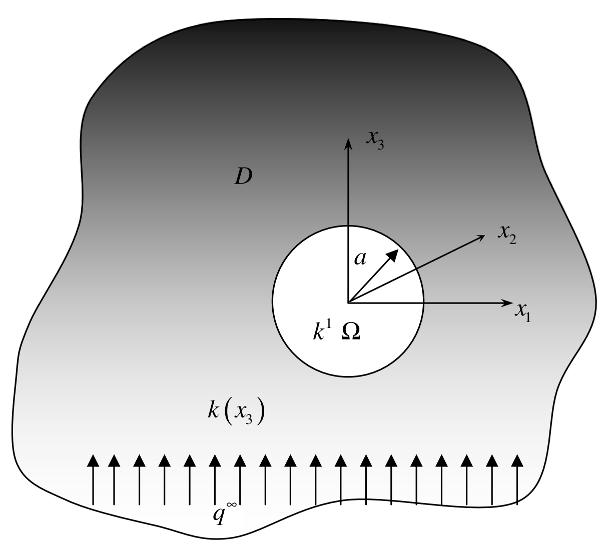

Fig. 2. A single spherical inhomogeneity in an FGM matrix subjected to a uniform heat flux field.

$$q^{\infty}\delta_{i3} + q'_i(\mathbf{x}) = q^*(\mathbf{x})\delta_{i3} - k(x_3)T_{,i}(\mathbf{x}) \quad \mathbf{x}\in\Omega, \tag{20}$$

where the prescribed heat flux is approximated by the linear function in (13), and the heat flux disturbance in the particle domain due to the prescribed heat flux can be obtained from (12) as

$$\begin{aligned} q'_i(\mathbf{x}) = {} & (q^0 + \tilde{q}x_3)\delta_{i3} - \frac{1+\alpha x_3}{30} \\ & \times \Big\{ q^0(10\delta_{i3} + 10\alpha x_i) + 6\left(\tilde{q} - \alpha q^0\right)(2\delta_{i3}x_3 + x_i) \\ & -\alpha\left(\tilde{q} - 2\alpha q^0\right)\left[(5a^2 - 3|\mathbf{x}|^2)\delta_{i3} - 6x_3x_i\right] \Big\} \\ & + \frac{\alpha\delta_{i3}}{30}\Big\{ q^0\left[10x_3 - 5\alpha(3a^2 - |\mathbf{x}|^2)\right] \\ & - \left(\tilde{q} - \alpha q^0\right)\left[\left(5a^2 - 3|\mathbf{x}|^2\right) - 6x_3^2\right] \\ & - \alpha\left(\tilde{q} - 2\alpha q^0\right)\left(5a^2 - 3|\mathbf{x}|^2\right)x_3 \Big\}. \end{aligned} \tag{21}$$

On the other hand, because the thermal property of the particle is replaced by the corresponding FMG properties in (4), the heat flux of the real particle should be equal to that of the equivalent inclusion, so that the combination of (19) and (20) yields:

$$q^*(\mathbf{x}) = \frac{k^1 - k(x_3)}{k^1}\left[q^{\infty} + q'_3(\mathbf{x})\right]. \tag{22}$$

Here q^* is written in the form of (13). By manipulating the right hand side of (22) using a Taylor series expansion applied at the origin, and by comparing the coefficients up to the linear terms, the following two expressions are obtained:

$$\begin{aligned} q^0 &= \left(1 - \frac{k^0}{k^1}\right)\left[q^{\infty} + \frac{2q^0}{3}\left(1 - a^2\alpha^2\right)\right], \\ \tilde{q} &= \left(1 - \frac{k^0}{k^1}\right)\frac{4\alpha q^0 + 6\tilde{q}}{15} - \frac{2\alpha k^0}{k^1}\left[q^{\infty} + \frac{2q^0}{3}\left(1 - a^2\alpha^2\right)\right]. \end{aligned} \tag{23}$$

From the above two equations, q^0 and $\tilde{q}$ can be explicitly written as

$$\begin{aligned} q^0 &= \frac{k^1 - k^0}{3k^1 - 2(1 - a^2\alpha^2)\left(k^1 - k^0\right)}3q^{\infty}; \\ \tilde{q} &= \frac{2\left(k^1 - k^0\right)^2 - 15k^0k^1}{\left(3k^1 + 2k^0\right)\left[3k^1 - 2(1 - a^2\alpha^2)\left(k^1 - k^0\right)\right]}2\alpha q^{\infty}. \end{aligned} \tag{24}$$

Using the assumption of (5), one can disregard the higher order term αa. Truncating the terms αa in (24) up to linear term, one can write the explicit solution of q^0 and $\tilde{q}$ as

$$\begin{aligned} q^0 &= \frac{k^1 - k^0}{k^1 + 2k^0}3q^{\infty}; \\ \tilde{q} &= \frac{2\left(k^1 - k^0\right)^2 - 15k^0k^1}{\left(3k^1 + 2k^0\right)\left(k^1 + 2k^0\right)}2\alpha q^{\infty}. \end{aligned} \tag{25}$$

In summary, the heat flux field can be explicitly written as follows:

$$q_i(\mathbf{x}) = q^{\infty}\delta_{i3} + q^*(\mathbf{x})\delta_{i3} - k^0(1 + \alpha x_3)U_{,i}(\mathbf{x}) + k^0\alpha\delta_{i3}U(\mathbf{x}), \tag{26}$$

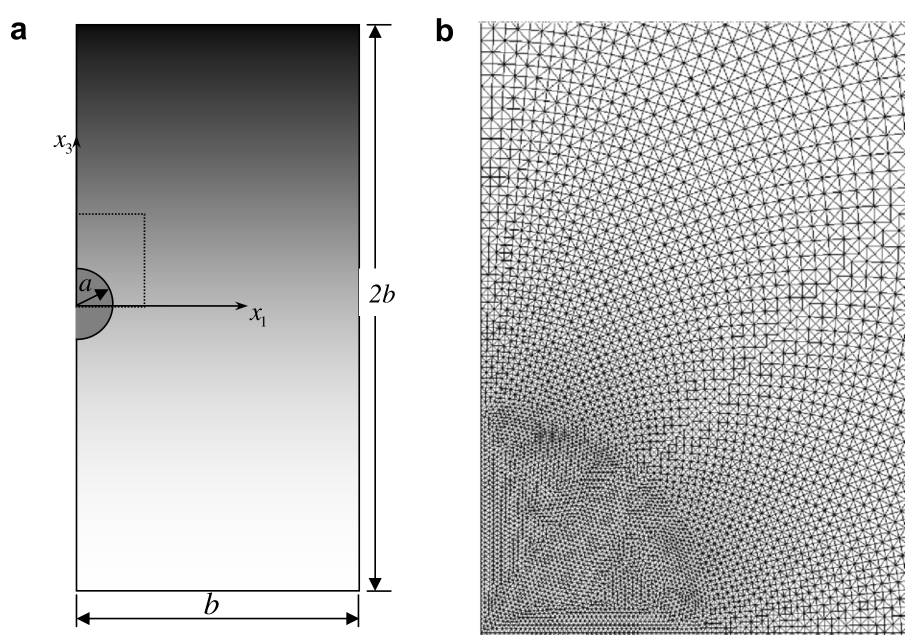

Fig. 3. Finite element model for a single inhomogeneity embedded in a large functionally graded material: (a) axisymmetric geometry, and (b) finite element mesh in the dotted line box.

where

$$q^*(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \notin \Omega, \\ q^0 + \tilde{q}x_3, & \mathbf{x} \in \Omega, \end{cases} \tag{27}$$

$$U(\mathbf{x}) = \begin{cases} \frac{1}{15k^0}\left[\begin{array}{l} 5\rho a q^0(\rho n_3 - 5\alpha a) - \rho^3 a^2\left(1-3n_3^2\right)(\tilde{q}-\alpha q^0) \\ -\alpha\rho^2 a^3 n_3(\tilde{q}-2\alpha q^0) \end{array}\right], & \mathbf{x} \notin \Omega, \\ \frac{1}{30k^0}\left[\begin{array}{l} q^0\left(10x_3 - 5\alpha\left(3a^2 - |\mathbf{x}|^2\right)\right) - (\tilde{q}-\alpha q^0)\left(5a^2 - 3|\mathbf{x}|^2 - 6x_3^2\right) \\ -\alpha(\tilde{q}-2\alpha q^0)\left(5a^2-3|\mathbf{x}|^2\right)x_3 \end{array}\right], & \mathbf{x} \in \Omega, \end{cases} \tag{28}$$

$$U_{,i}(\mathbf{x}) = \begin{cases} \frac{1}{15k^0}\left[\begin{array}{l} 5q^0\rho^2(\rho(\delta_{i3} - 3n_3 n_i) + \alpha a n_i) + 3(\tilde{q}-\alpha q^0)\rho^4 a\left(2\delta_{i3}n_3 + n_i - 5n_3^2 n_i\right) \\ -\alpha(\tilde{q}-2\alpha q^0)\rho^3 a^2(\delta_{i3} - 3n_3 n_i) \end{array}\right], & \mathbf{x} \notin \Omega, \\ \frac{1}{30k^0}\left(\begin{array}{l} 10q^0(\delta_{i3} + \alpha x_i) + 6(\tilde{q}-\alpha q^0)(2\delta_{i3}x_3 + x_i) \\ -\alpha(\tilde{q}-2\alpha q^0)\left[\left(5a^2 - 3|\mathbf{x}|^2\right)\delta_{i3} - 6x_3 x_i\right] \end{array}\right), & \mathbf{x} \in \Omega \end{cases} \tag{29}$$

in which $\alpha = 0.5k'(0)/k^0$, and q^0 and $\tilde{q}$ are provided in (25).

4. Model verification and discussion

The thermal fields for a single inhomogeneity embedded in a functionally graded material are of great interests for thermal analysis of FGM structures. For example, during fabrication of FGMs, air voids may form and change the thermomechanical behavior of the FGMs during their application. Because the present solution of the thermal fields for a particle embedded in an infinitely large FGM involves linear approximations, the accuracy of the solution must be investigated. Although numerical methods cannot provide exact solution for the thermal fields in FGMs [7–9,20–22], a good approximation of the thermal field in FGMs can be reached by using a highly refined discretization with graded material property.

Herein, a finite element model is constructed with the finite element software ABAQUS to offer a reference solution. Based on the geometry and loading conditions, an axisymmetric problem is considered as shown in Fig. 3a. Here the size of the FGM is $b \times 2b$, and the radius of the inhomogeneity is a. Because the disturbed heat flux field is mostly influenced by the material distribution in the neighborhood of the particle, herein the FGM size $b = 10a$ is used, which was found to provide a convergent solution for the local heat flux field in this problem. The radius of the particle is assumed to be of the unit length, i.e. $a = 1$ m. Triangular elements are employed, where the edge length of the elements in the particle is 0.02, whereas those for the FGM are mostly 0.1. However, in the neighborhood of the particle, a transition zone is used with gradually changing element size. Fig. 3b illustrates the finite element mesh in the region of the box denoted by dotted lines in Fig. 3a. The mesh consists of 71,060 elements and 36,935 nodes. The material properties of each element are assumed to be uniform and depend on the location of the centroid of the element. Herein the user-defined subroutine UMATHT [18] is employed to assign material properties to each element. Although each element has uniform material properties, the sizes of the elements are relatively small with respect to the characteristic length ($1/\alpha$) associated to the material gradient [19]. Thus, the present mesh with varying material properties should be sufficient to represent the gradient of the FGM.

First, one air void embedded in an FGM, which is subjected to a uniform heat flux field, is investigated. The heat conductivity of air is taken as $k^1 = 0.03$ W/m K. For the FGM, we assume $k^0 = 1.0$ W/m K and $\alpha = 0.05$ m^{-1}. A uniform heat flux field $q_3 = 1.0$ W/m^2 is applied along the bottom and surface of the FGM, such that a steady-state heat flux field is induced but the local heat flux field is disturbed by the material mismatch between the air void and the FGM. Fig. 4 compares the heat flux distribution along the axes x_1 and x_3 for the numerical (FEM) and analytical solutions. Along the axis x_1, the magnitude of heat flux field is nearly uniform in the particle domain but highly discontinuous across the interface. The heat flux reaches its maximum at the outer surface of the particle and then gradually decreases to 1.0 W/m^2 in the far field of the matrix material. However, along the axis x_3, the distribution of heat flux field is nearly linear in the particle domain and continuous even across the interface. Far from the particle, the heat flux is also convergent to 1.0 W/m^2, which is the uniform far field loading. The proposed solution is in excellent agreement with the FEM results. From the heat flux field in the particle, we can see that the linear assumption of heat flux distribution in the gradation direction in the particle domain is quite sufficient. The heat flux field is almost convergent to 1.0 W/m^2 at the distance of four times of the particle's radius, whereas the finite element mesh used herein covers a region spanning 10 times the particle's radius. Therefore, the domain extent of the currently used mesh is adequate.

Next, a parametric investigation conducted using the same finite element model. First, we consider the effect of

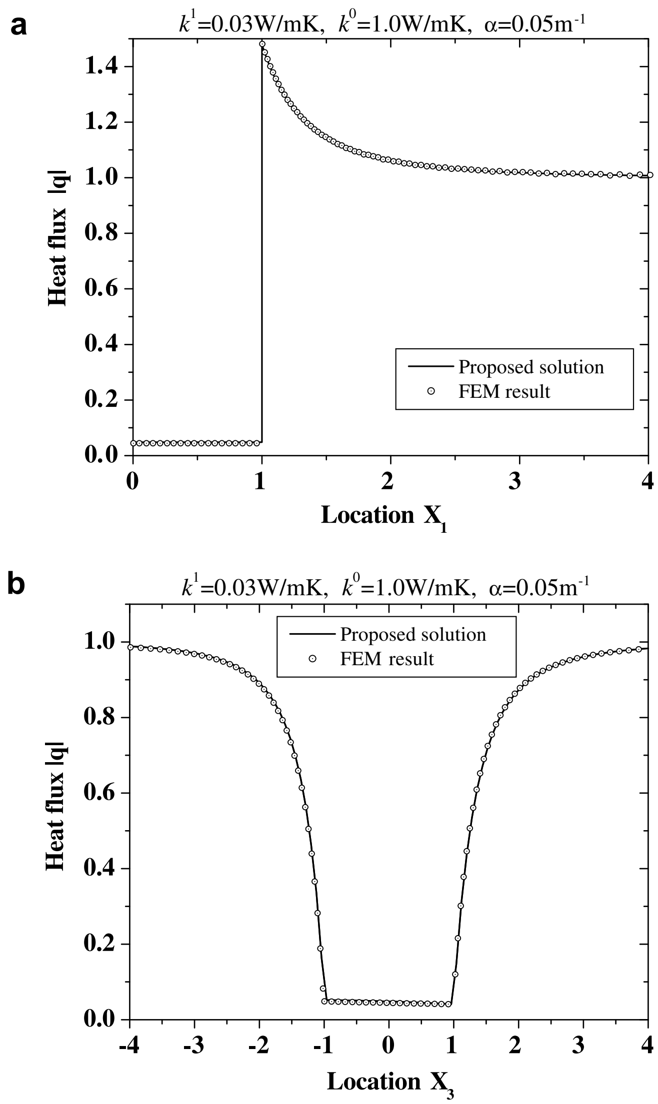

Fig. 4. Comparison of heat flux distribution for an air void embedded in a graded material for the proposed solution and the finite element result: (a) along the x_1 axis and (b) along the x_3 axis.

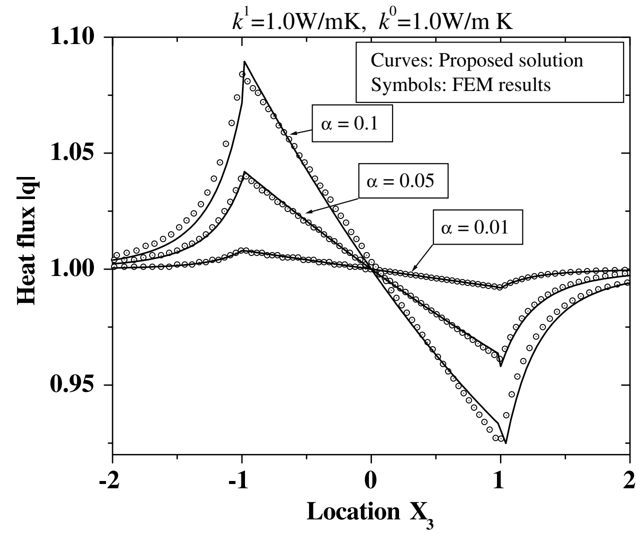

Fig. 5. Effect of matrix material gradation on the heat flux distribution along the x_3 axis.

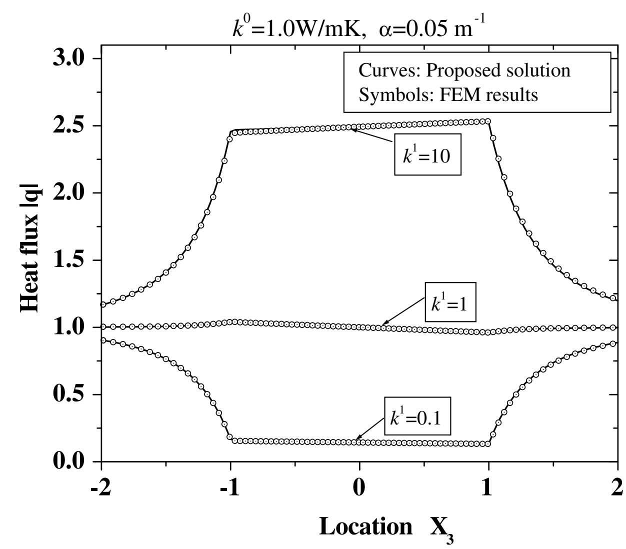

Fig. 6. Effect of the particle's thermal conductivities on the heat flux distribution along the x_3 axis.

material gradation. The thermal conductivities for the particle and the graded material are taken as $k^1 = 1.0$ W/m K and $k^0 = 1.0$ W/m K. Three material variations are investigated, namely $\alpha = 0.01$, 0.05, and 0.1 m^{-1}. As shown in Fig. 5, the first two variations adequately satisfy the condition described in (5) and thus provide solutions which are in very good agreement with the FEM results; whereas the last variation is quite high, which leads to minor discrepancies with the numerical solution. Because the proposed method uses linear assumptions and truncates the effect of higher order terms, it provides excellent prediction when α is relatively small. However, for a large material gradient, such as $\alpha a > 0.1$, the accuracy of the solution begins to degrade.

Fig. 6 investigates the effect of the material mismatch between particles and the surrounding graded material on the heat flux distribution. The material parameters for the graded material are taken as $k^0 = 1.0$ W/m K and $\alpha = 0.05\ \mathrm{m}^{-1}$, whereas three particle thermal conductivities are investigated: $k^1 = 10.0$, 1.0, and 0.1 W/m K. Fig. 6 illustrates that excellent agreement of the proposed solution to the FEM results are obtained for all three cases, so the accuracy of the proposed solution is highly insensitive to the material mismatch.

5. Conclusions

The present study provides an explicit formulation of the heat flux distribution for a single inhomogeneity embedded in an unbounded graded material. Green's function technique is employed to solve the heat flux field due to a prescribed heat flux field in a particle domain, which is embedded in an unbounded graded material. Using the equivalent inclusion method, the particle-graded matrix is transferred to a homogeneous graded material domain but with a prescribed heat flux field acting in the particle domain. The heat flux field is analytically derived. The present solution is in excellent agreement with finite element results when material property gradients are relatively small. When particle's size is much smaller than the overall size of an FGM, this solution is applicable to solving for particle's heat flux for any continuous and differentiable material variation based on the linearization assumption. In addition, this work can naturally extend to other areas described by equations of potential theory, such as for electric, dielectric, magnetic, and water flow problems.

Acknowledgements

This work is sponsored by Federal Highway Administration National Pooled Fund Study 776, whose support is gratefully acknowledged. The results and opinions presented herein are those of the authors and do not necessarily reflect those of the sponsoring agency.

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