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Correspondence Principle in Viscoelastic Functionally Graded Materials

G. H. Paulino[1]
Mem. ASME
e-mail: paulino@uiuc.edu

Z.-H. Jin
Mem. ASME

Department of Civil and Environmental Engineering,
University of Illinois at Urbana-Champaign,
Urbana, IL 61801

This paper presents an extension of the correspondence principle (as applied to homogeneous viscoelastic solids) to nonhomogeneous viscoelastic solids under the assumption that the relaxation (or creep) moduli be separable functions in space and time. A few models for graded viscoelastic materials are presented and discussed. The revisited correspondence principle extends to specific instances of thermoviscoelasticity and fracture of functionally graded materials. [DOI: 10.1115/1.1331286]

1 Introduction

Functionally graded materials (FGMs) are special composites usually made from both ceramics and metals. The ceramic in an FGM offers thermal barrier effects and protects the metal from corrosion and oxidation. The FGM is toughened and strengthened by the metallic composition. *The composition and the volume fraction of the constituents vary gradually, giving a nonuniform microstructure with continuously graded macroproperties*. Various thermomechanical problems of FGMs have been studied, for example, constitutive modeling ([1]), fracture behavior ([2–4]), thermal stresses ([5,6]), strain gradient effects ([7]), plate bending problems ([8]), higher order theory ([9]), and so on. Comprehensive reviews of ongoing FGM research may be found in the article by Hirai [10] and the book by Suresh and Mortensen [11].

One of the primary application areas of FGMs is high-temperature technology. Materials will exhibit creep and stress relaxation behavior at high temperatures. Viscoelasticity offers a basis for the study of phenomenological behavior of creep and stress relaxation. The elastic-viscoelastic correspondence principle (or elastic-viscoelastic analogy) is probably one of the most useful tools in viscoelasticity because the Laplace transform of the viscoelastic solution can be directly obtained from the corresponding elastic solution. In the present work, the correspondence principle is revisited in the context of viscoelastic FGMs.

In this paper, the basic equations of viscoelasticity in FGMs are formulated. The correspondence principle is established for a class of FGMs where the relaxation moduli for shear and dilatation $\mu(\mathbf{x},t)$ and $K(\mathbf{x},t)$ take the forms $\mu(\mathbf{x},t)=\mu_0\tilde{\mu}(\mathbf{x})f(t)$ and $K(\mathbf{x},t)=K_0\tilde{K}(\mathbf{x})g(t)$, respectively, where μ_0 and K_0 are material constants, $\tilde{\mu}(\mathbf{x})$, $\tilde{K}(\mathbf{x})$, $f(t)$, and $g(t)$ are nondimensional functions, and $\mathbf{x}=(x_1,x_2,x_3)$. The correspondence principle states that the Laplace transforms of the nonhomogeneous viscoelastic variables can be obtained from the nonhomogeneous elastic variables by replacing μ_0 and K_0 with $\mu_0 p\bar{f}(p)$ and $K_0 p\bar{g}(p)$, respectively, where $\bar{f}(p)$ and $\bar{g}(p)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, and p is the transform variable. The final nonhomogeneous viscoelastic solution is realized by inverting the transformed solution. The above correspondence principle can also be extended to specific instances of thermoviscoelasticity and fracture of FGMs.

2 Basic Equations

The basic equations of quasi-static viscoelasticity of FGMs are the equilibrium equation

$$\sigma_{ij,j}=0, \tag{1}$$

the strain-displacement relationship

$$\epsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \tag{2}$$

and the viscoelastic constitutive law

$$s_{ij}=2\int_0^t \mu(\mathbf{x},t-\tau)\frac{de_{ij}}{d\tau}d\tau, \quad \sigma_{kk}=3\int_0^t K(\mathbf{x},t-\tau)\frac{d\epsilon_{kk}}{d\tau}d\tau, \tag{3}$$

in which σ_{ij} are stresses, ϵ_{ij} are strains, s_{ij} and e_{ij} are deviatoric components of stress and strain tensors given by

[1]To whom correspondence should be addressed.

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$$s_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}, \quad e_{ij}=\epsilon_{ij}-\frac{1}{3}\epsilon_{kk}\delta_{ij}, \tag{4}$$

where u_i are displacements, δ_{ij} is the Kronecker delta, $\mu(\mathbf{x},t)$ and $K(\mathbf{x},t)$ are appropriate relaxation functions, t is time, and the Latin indices have the range 1, 2, 3 with repeated indices implying the summation convention. Note that the relaxation functions also depend on spatial positions, whereas in homogeneous viscoelasticity, they are only functions of time, i.e., $\mu\equiv\mu(t)$ and $K\equiv K(t)$ ([12]).

For a boundary value problem, the boundary conditions are given by

$$\sigma_{ij}n_j=S_i, \quad \text{on } B_\sigma, \tag{5}$$

$$u_i=\Delta_i, \quad \text{on } B_u, \tag{6}$$

where n_j are the components of the unit outward normal to the boundary of the body, S_i are the tractions prescribed on B_σ, and Δ_i are the prescribed displacements on B_u. The parts of the boundary B_σ and B_u are required to remain constant with time.

3 Correspondence Principle

In general, the correspondence principle of homogeneous viscoelasticity may not hold for FGMs. To circumvent this problem, we consider a class of FGMs in which the relaxation functions have the following general form:

$$\begin{aligned}\mu(\mathbf{x},t)&=\mu_0\tilde{\mu}(\mathbf{x})f(t),\\ K(\mathbf{x},t)&=K_0\tilde{K}(\mathbf{x})g(t),\end{aligned} \tag{7}$$

where μ_0 and K_0 are material constants, and $\tilde{\mu}(\mathbf{x})$, $\tilde{K}(\mathbf{x})$, $f(t)$, and $g(t)$ are nondimensional functions. The constitutive law (3) is then reduced to

$$\begin{aligned}s_{ij}&=2\mu_0\tilde{\mu}(\mathbf{x})\int_0^t f(t-\tau)\frac{de_{ij}}{d\tau}d\tau,\\ \sigma_{kk}&=3K_0\tilde{K}(\mathbf{x})\int_0^t g(t-\tau)\frac{d\epsilon_{kk}}{d\tau}d\tau.\end{aligned} \tag{8}$$

By assuming the material initially at rest, the Laplace transforms of the basic Eqs. (1), (2), (8), and the boundary conditions (5) and (6) are obtained as

$$\bar{\sigma}_{ij,j}=0, \tag{9}$$

$$\bar{\epsilon}_{ij}=\frac{1}{2}(\bar{u}_{i,j}+\bar{u}_{j,i}), \tag{10}$$

$$\bar{s}_{ij}=2\mu_0\tilde{\mu}(\mathbf{x})p\bar{f}(p)\bar{e}_{ij}, \tag{11}$$

$$\bar{\sigma}_{kk}=3K_0\tilde{K}(\mathbf{x})p\bar{g}(p)\bar{\epsilon}_{kk}, \tag{12}$$

$$\bar{\sigma}_{ij}n_j=\bar{S}_i, \quad \text{on } B_\sigma, \tag{13}$$

$$\bar{u}_i=\bar{\Delta}_i, \quad \text{on } B_u, \tag{14}$$

where a bar over a variable represents its Laplace transform, and p is the transform variable. Thus

$$\begin{aligned}&\bar{\sigma}_{ij}=\int_0^\infty \sigma_{ij}\exp(-pt)dt, \quad \bar{\epsilon}_{ij}=\int_0^\infty \epsilon_{ij}\exp(-pt)dt,\\ &\bar{u}_i=\int_0^\infty u_i\exp(-pt)dt, \quad \bar{f}(p)=\int_0^\infty f(t)\exp(-pt)dt,\\ &\bar{g}(p)=\int_0^\infty g(t)\exp(-pt)dt.\end{aligned} \tag{15}$$

It is seen that the set of Eqs. (9)–(12), and conditions (13) and (14) have a form identical to those of nonhomogeneous elasticity with the shear modulus $\mu=\mu_0\tilde{\mu}(\mathbf{x})$ and the bulk modulus $K=K_0\tilde{K}(\mathbf{x})$ provided that the transformed viscoelastic variables are associated with the corresponding elastic variables and $\mu_0p\bar{f}(p)$ and $K_0p\bar{g}(p)$ are associated with μ_0 and K_0, respectively. Therefore, the *correspondence principle* in homogeneous viscoelasticity still holds for the FGM with the material properties given in Eq. (7), i.e., *the Laplace transformed nonhomogeneous viscoelastic solution can be obtained directly from the solution of the corresponding nonhomogeneous elastic problem by replacing μ_0 and K_0 with $\mu_0p\bar{f}(p)$ and $K_0p\bar{g}(p)$, respectively. The final solution is realized upon inverting the transformed solution.*

4 Some Models for Graded Viscoelastic Materials

Among the various models for graded viscoelastic materials are the *standard linear solid* defined by

$$\begin{aligned}\mu(\mathbf{x},t)&=\mu_\infty(\mathbf{x})+[\mu_e(\mathbf{x})-\mu_\infty(\mathbf{x})]\exp\left[-\frac{t}{t_\mu(\mathbf{x})}\right],\\ K(\mathbf{x},t)&=K_\infty(\mathbf{x})+[K_e(\mathbf{x})-K_\infty(\mathbf{x})]\exp\left[-\frac{t}{t_K(\mathbf{x})}\right],\end{aligned} \tag{16}$$

the *power-law model*

$$\mu(\mathbf{x},t)=\mu_e(\mathbf{x})\left[\frac{t_\mu(\mathbf{x})}{t}\right]^q, \quad K(\mathbf{x},t)=K_e(\mathbf{x})\left[\frac{t_K(\mathbf{x})}{t}\right]^q, \quad 0<q<1, \tag{17}$$

and the *Maxwell material*

$$\mu(\mathbf{x},t)=\mu_e(\mathbf{x})\exp\left[-\frac{t}{t_\mu(\mathbf{x})}\right], \quad K(\mathbf{x},t)=K_e(\mathbf{x})\exp\left[-\frac{t}{t_K(\mathbf{x})}\right], \tag{18}$$

where $t_\mu(\mathbf{x})$ and $t_K(\mathbf{x})$ are the relaxation times in shear and bulk moduli, respectively, and q is a material constant. The discussion below indicates the revisions needed in the general models so that the correspondence principle holds.

- Standard Linear Solid (16). If the relaxation times t_μ and t_K are constant, if $\mu_e(\mathbf{x})$ and $\mu_\infty(\mathbf{x})$ have the same functional form, and if $K_e(\mathbf{x})$ and $K_\infty(\mathbf{x})$ have the same functional form, then the standard linear solid satisfies assumption (7).
- Power Law Model (17). It is seen that if the relaxation times t_μ and t_K are independent of spatial position, then the assumption (7) is readily satisfied. Moreover, even if the relaxation times depend on the spatial position in (17), the correspondence principle may still be applied with some revision, which consists of taking the corresponding nonhomogeneous elastic material with the following properties:

$$\mu=\mu_e(\mathbf{x})[t_\mu(\mathbf{x})]^q, \quad K=K_e(\mathbf{x})[t_K(\mathbf{x})]^q, \tag{19}$$

instead of $\mu=\mu_e(\mathbf{x})$ and $K=K_e(\mathbf{x})$.

- Maxwell Material (18). If the relaxation times t_μ and t_K are independent of spatial position, the assumption (7) is promptly satisfied.

5 Thermoviscoelastic Problem

The basic equations of thermoviscoelasticity of FGMs are identical to those of viscoelasticity except the constitutive law. The constitutive relation for thermoviscoelastic FGMs is given by

$$\begin{aligned}s_{ij}&=2\int_0^t \mu(\mathbf{x},t-\tau)\frac{de_{ij}}{d\tau}d\tau,\\ \sigma_{kk}&=3\int_0^t K(\mathbf{x},t-\tau)\frac{d[\epsilon_{kk}-\alpha(\mathbf{x})T]}{d\tau}d\tau,\end{aligned} \tag{20}$$

where T is the temperature and $\alpha(\mathbf{x})$ is the coefficient of thermal expansion. Here α is assumed to be time-independent. By applying the Laplace transform to the above equation and adopting the form of the relaxation functions given in (7), we obtain

$$\overline{s_{ij}}=2\mu_0\tilde{\mu}(\mathbf{x})p\overline{f(p)}\overline{e_{ij}},\quad \overline{\sigma_{kk}}=3K_0\tilde{K}(\mathbf{x})p\overline{g}(p)(\overline{\epsilon_{kk}}-\alpha\overline{T}), \tag{21}$$

while the constitutive relation of the nonhomogeneous thermoelasticity may be expressed as

$$s_{ij}=2\mu_0\tilde{\mu}(\mathbf{x})e_{ij},\quad \sigma_{kk}=3K_0\tilde{K}(\mathbf{x})(\epsilon_{kk}-\alpha T). \tag{22}$$

Thus it can be seen that the correspondence principle still holds.

6 A Path-Independent Integral

The J-integral ([13]) has been extended to certain classes of elastic materials with varying Young's modulus in the crack-line direction by Honein and Herrmann [14]. Here, a J-like path-independent integral is presented for characterizing fracture in nonhomogeneous viscous materials.

Consider the shear modulus with the specific functional form

$$\mu(x_1,x_2,t)=\mu_0(x_2)\exp(\beta x_1)f(t) \tag{23}$$

where $\mu_0(x_2)$ is an arbitrary function of x_2 and β is an arbitrary material constant. Note that (23) has the form given in (7). Moreover, the Poisson's ratio is assumed to be independent of x_1. The proposed integral to characterize crack growth in such graded material undergoing creep is

$$C_e^*=\int_\Gamma\left[\left(\dot{W}n_1-\sigma_{ij}n_j\frac{\partial\dot{u}_i}{\partial x_1}\right)-\frac{\beta}{2}\sigma_{ij}n_j\dot{u}_i\right]ds \tag{24}$$

where Γ is a contour enclosing the crack tip, n_1 is the first component of the unit outward normal to Γ, $\sigma_{ij}n_j=S_i$ are the components of tractions along Γ, ds is an infinitesimal length element along the contour Γ, and $\dot{W}$ is the stress work rate (power) density defined as

$$\dot{W}=\int_0^{\dot{\epsilon}_{kl}}\sigma_{ij}d\dot{\epsilon}_{ij}. \tag{25}$$

The integral (24) has been obtained by replacing strain with strain rates, and displacement with displacement rates in the corresponding J_e-integral ([14]) for nonhomogeneous elastic materials.

The integral of the term within parentheses in (24) is the so-called C^* integral (e.g., [15]) which is valid for homogeneous viscous materials undergoing steady-state creep. The extra term in (24), which appears outside the parentheses, is due to the modulus variation. Equation (24) can be seen as an extension of the C^* integral for nonhomogeneous viscous media. The C^* integral is a special case of the J_v-integral derived by Schapery [16] by means of correspondence principle arguments. The latter integral accounts for a wide range of time-dependent material behavior, and includes viscous creep as special case.

7 A Simple Example

As an example of application, we consider an infinite strip of width h occupying the region $0\leqslant x_1\leqslant h$, $-\infty<x_2<\infty$, $-\infty<x_3<\infty$. It is assumed that the strip deforms in the x_1-x_2 plane under the plane-strain conditions. A "fixed grip" loading condition is considered, i.e., $\epsilon_{22}(x_1,\pm\infty)=\epsilon_0$, where ϵ_0 is a constant. The nonvanishing stress σ_{22} in a nonhomogeneous elastic material with the Young's modulus $E=E_e(x_1)$ and the Poisson's ratio $\nu=\nu_e(x_1)$ is given by ([4])

$$\sigma_{22}=\frac{E_e(x_1)\epsilon_0}{1-\nu_e^2(x_1)}=\frac{4\epsilon_0\mu_e(x_1)[3K_e(x_1)+\mu_e(x_1)]}{3K_e(x_1)+4\mu_e(x_1)}, \tag{26}$$

where the following relations are used:

$$E_e=\frac{9K_e\mu_e}{3K_e+\mu_e},\quad \nu_e=\frac{3K_e-2\mu_e}{2(3K_e+\mu_e)}. \tag{27}$$

According to the correspondence principle, the Laplace transform of the stress in a viscoelastic FGM with the shearing and dilatational relaxation functions $\mu=\mu_e(x_1)f(t)$ and $K=K_e(x_1)g(t)$ is given by

$$\overline{\sigma_{22}}=\frac{4\epsilon_0\mu_e(x_1)\overline{f}(p)[3K_e(x_1)\overline{g}(p)+\mu_e(x_1)\overline{f}(p)]}{3K_e(x_1)\overline{g}(p)+4\mu_e(x_1)\overline{f}(p)}. \tag{28}$$

For the Maxwell material (18) with constant relaxation times t_μ and t_K, the above transformed stress becomes

$$\overline{\sigma_{22}}=\frac{[4\epsilon_0\mu_e(x_1)/(p+1/t_\mu)][3K_e(x_1)/(p+1/t_K)+\mu_e(x_1)/(p+1/t_\mu)]}{3K_e(x_1)/(p+1/t_K)+4\mu_e(x_1)/(p+1/t_\mu)}. \tag{29}$$

By inverting (29), we get the stress in the time domain as follows:

$$\sigma_{22}=\left\{\frac{9K_e(x_1)}{4\mu_e(x_1)+3K_e(x_1)}\exp\left[-\frac{4\mu_e(x_1)t_\mu/t_K+3K_e(x_1)}{4\mu_2(x_1)+3K_e(x_1)}\frac{t}{t_\mu}\right]+\exp\left(-\frac{t}{t_\mu}\right)\right\}\mu_e(x_1)\epsilon_0. \tag{30}$$

By letting $t\to 0^+$, the nonhomogeneous elastic solution (26) is recovered.

8 Conclusions

The correspondence principle is revisited and established for a class of FGMs where the relaxation functions for shear and dilatation take separable forms in space and time, i.e., $G_1(\mathbf{x},t)/2=\mu(\mathbf{x},t)=\mu_0\tilde{\mu}(\mathbf{x})f(t)$ and $G_2(\mathbf{x},t)/3=K(\mathbf{x},t)=K_0\tilde{K}(\mathbf{x})g(t)$, respectively. The correspondence principle states that the Laplace transforms of the nonhomogeneous viscoelastic variables can be obtained from the nonhomogeneous elastic variables by replacing μ_0 and K_0 with $\mu_0p\overline{f}(p)$ and $K_0p\overline{g}(p)$, respectively, where $\overline{f}(p)$ and $\overline{g}(p)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, and p is the transform variable. The final nonhomogeneous viscoelastic solution is realized by inverting the transformed solution. Equivalently, if the creep functions $J_1(\mathbf{x},t)$ and $J_2(\mathbf{x},t)$ have separable forms in space and time, then the correspondence principle (as employed here) is also directly applicable.

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On Some Anomalies in Lamé's Solutions for Elastic Solids With Holes

G. B. Sinclair
Department of Mechanical Engineering, Louisiana State University, Baton Rouge, LA 70803-6413

G. Meda
Science and Technology Division, Corning, Inc., Corning, NY 14831-0001

Elastic solids with holes under remote tension are reconsidered. When hole dimensions are shrunk so that holes disappear, anomalies occur in the classical elasticity solutions of Lamé. By introducing cohesive laws on hole surfaces as they shrink, these anomalies may be removed. [DOI: 10.1115/1.1331285]

1 The Issue

Sketched in Fig. 1 is the Lamé problem of an infinite elastic plate, weakened by a circular hole of radius a, under a uniform remote tension σ_0. In cylindrical polar coordinates (Fig. 1), the stresses in its classical solution are given in Lamé [1] and are

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = \sigma_0 \left(1 \begin{Bmatrix} - \\ + \end{Bmatrix} \frac{a^2}{r^2} \right), \tag{1}$$

for $a \leqslant r < \infty$, $0 \leqslant \theta < 2\pi$. The companion shear stress component is zero by virtue of the axisymmetry of the configuration. That such a stress field is indeed a valid solution within classical elasticity can be verified by direct substitution into the governing field equations and the boundary conditions.

Setting $r=a$ in σ_θ of (1) reveals a stress concentration factor (SCF) of 2 at the edge of the hole. Consider what happens to this concentration factor if $a \to 0$ and the hole disappears. The SCF is independent of a, so it remains equal to 2 even when $a \to 0$. This is inconsistent with what one would expect physically, namely that the limit $a \to 0$ should be the same as when the plate is whole without a hole and has no stress concentration.

The same sort of anomalous result occurs for an elastic solid with a spherical hole. Then Lamé [1] has that the SCF is 3/2 independent of the hole radius. Again, therefore, there is a stress concentration when the radius goes to zero, inconsistent with physical expectations.

These anomalous results are passed by without comment in Lamé [1]. While they have no doubt been noted by elasticans since, their existence may well not be as widely appreciated today as it could be. They are not mentioned in classical texts which include the Lamé solutions (e.g., Love [2], Art. 100, 98; Muskhelishvili [3], Art. 56a; Timoshenko and Goodier [4], Art. 28, 136). Further, we could not find them discussed in any other standard elasticity texts.[1] Nevertheless they bear explaining.

Mathematically, there is a clear distinction between solids with holes with radii tending to zero and solids without holes. When $a \to 0$ in either of Lamé's hole problems, the boundary condition $\sigma_r = 0$ holds at $r=0$. In contrast, for a plate without a hole, the field equations hold at $r=0$. Such mathematical distinctions, however, fall short of a fully satisfactory physical explanation.

We have been offered physical explanations of the following genre by a number of people: ''Physically speaking, one explains this nonuniform behavior as the presence of a stress concentration in an imperfect body such as at the boundary of a small entrained cavity in a casting.'' To examine the physical appropriateness of such explanations, we consider a further limit of (1) as $a \to 0$. Specifically, we take σ_θ of (1) at $\theta = 0$, denote it by $\hat{\sigma}_y$, and set $r = \lambda a$, $\lambda \geqslant 1$. Then

$$\hat{\sigma}_y = \sigma_0 (1 + \lambda^{-2}) \quad \text{as} \quad a \to 0. \tag{2}$$

Of course, as $a \to 0$, $r = \lambda a \to 0$ for all $\lambda \geqslant 1$. Hence with this model of an imperfection, as a tends to zero we can get any value

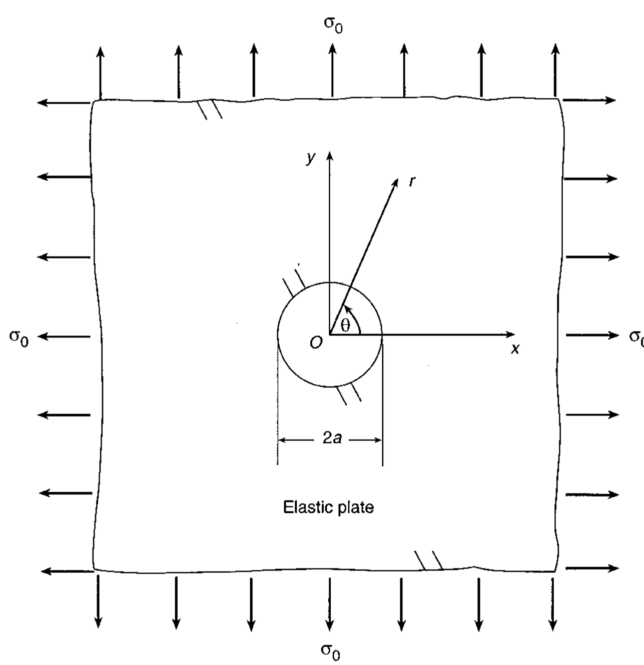

Fig. 1 Plate with hole under remote tension

[1]Swain [5], pp. 121,122, does note a similar anomalous result in the classical elasticity solution for an infinite plate with a circular hold under *uniaxial* far-field tension.

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of $\hat{\sigma}_y$ at $r=0$ between σ_0 and $2\sigma_0$ as the imperfection's stress concentration.[2] This unsatisfactory situation is compounded by the ambiguity of which stress component σ_x or σ_y is what in the limit as $a\to 0$ for different θ. All told, such physical explanations are quite superficial. Here, then, we seek to furnish a physically sensible resolution of the differences between Lamé's solutions for plates with holes and responses for whole plates.

2 A Resolution

What is missing in the classical statement of Lamé's hole problems is the recognition that atoms or molecules on opposite sides of any hole must start to interact with each other as the hole closes. This interaction produces *cohesive stresses* on the hole walls. Here we model the action of these cohesive stresses.

There are three key elements in our simple models. First, we introduce cohesive stresses via cohesive stress-separation laws on hole boundaries. This simplifies the incorporation of the underlying solid-state physics and reduces the analysis of our models to involving just continuum mechanics. Such an approach was first introduced in Barenblatt [6] and has seen extensive use since (Sinclair [7] provides a recent bibliography). For the most part, it has been employed in the analysis of cracks, although Levy [8,9] treats a rigid inclusion without a crack. The implementation of cohesive stress-separation laws here could be viewed as the dual of their use in Levy [8,9].

Second, we only consider that portion of the cohesive stress-separation law near the equilibrium position. That is, we only track the action of cohesive stresses when the hole is extremely small. In this range, cohesive stress-separation laws can be taken as linear. Moreover, the constant of proportionality can be backed out by insisting that the insertion of such a cohesive law within the continuum without any hole leaves response there unaltered—a kind of cohesive-law patch test. For the present problem, this insertion is actually carried out on a circular ring of radius R in an elastic plate with the same moduli as the original plate. Then such a patch test in effect accounts for the action of all the atoms external to R on all those internal, and vice versa. Again, simplification is the intent. The so-simplified treatment does nonetheless serve to demonstrate the basic physics involved.[3]

Third, we take our cohesive stress-separation law as acting between the centers of the atoms or molecules comprising the hole surfaces: By symmetry, these atoms or molecules are diametrically opposed. The consequence of this assumption is that holes close when their radii reduce to half of the equilibrium center-to-center spacing of the atoms or molecules. This removes any ambiguity associated with $a\to 0$.

The corresponding reformulation of Lamé's problem for the plate with a hole then is as follows. Throughout the plate of Fig. 1 when a is small, we seek the axisymmetric planar stresses σ_r, σ_θ, and their companion displacement u_r, as functions of r, satisfying the following requirements: the stress equation of equilibrium in the absence of body forces,

$$r\sigma_{r,r}+\sigma_r-\sigma_\theta=0, \tag{3}$$

for $a<r<\infty$, $0\leqslant\theta<2\pi$; the stress-displacement relations for a homogeneous and isotropic, linear elastic solid,

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\end{Bmatrix}=\mu\left[\frac{3-\kappa}{\kappa-1}\Theta+2\begin{Bmatrix}u_{r,r}\\ r^{-1}u_r\end{Bmatrix}\right],\quad \Theta=u_{r,r}+r^{-1}u_r, \tag{4}$$

for $a<r<\infty$, $0\leqslant\theta<2\pi$, wherein Θ is the dilation, μ is the shear modulus and κ is $3-4\nu$ for plane strain, $(3-\nu)/(1+\nu)$ for plane stress, ν being Poisson's ratio; the cohesive stress-separation law on the hole boundary,

$$\sigma_r=k(2u_r+2a-\delta)\ \text{ at }\ r=a, \tag{5}$$

for $0\leqslant\theta<2\pi$, wherein k is the law stiffness and δ is the equilibrium separation of the atoms or molecules comprising the plate; and the condition applying the tension at infinity,

$$\sigma_r=\sigma_0\ \text{ as }\ r\to\infty, \tag{6}$$

for $0\leqslant\theta<2\pi$. In addition, from our cohesive-law patch test at $r=R$, we have $\sigma_r=k[u_r(r=R+\delta/2)-u_r(r=R-\delta/2)]$, leading to

$$k=4\mu/\delta(\kappa-1). \tag{7}$$

This is the value of the stiffness to be used in (5) when a is sufficiently small.

Solution of the problem in (3)–(6) is elementary and gives

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\end{Bmatrix}=\sigma_0\begin{Bmatrix}-\\ +\end{Bmatrix}\sigma_0'\frac{a^2}{r^2},\quad u_r=\frac{1}{4\mu}\left[\sigma_0 r(\kappa-1)+2\sigma_0'\frac{a^2}{r}\right], \tag{8}$$

where

$$\sigma_0'=\sigma_0-k\,\frac{2\mu(2a-\delta)+(\kappa+1)\sigma_0 a}{2(\mu+ka)}. \tag{9}$$

Observe that (8) and (9) recover Lamé's solution (1) when $k=0$, as they should.

Now consider what happens if the hole disappears. Introducing k of (7) into (9), and taking $a\to\delta/2$ to close the hole, gives $\sigma_0'=0$. Thus from (8),

$$\sigma_r=\sigma_\theta=\sigma_0\ \text{ as }\ a\to\delta/2. \tag{10}$$

Equation (10) is the physically sensible result for a plate without a hole.

A similar reformulation and analysis for the spherical hole problem leads to

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\end{Bmatrix}=\sigma_0\begin{Bmatrix}-\\ +1/2\end{Bmatrix}\sigma_0''\frac{a^3}{r^3}, \tag{11}$$

where

$$\sigma_0''=\sigma_0-k\,\frac{4\mu(2a-\delta)+3(\kappa-1)\sigma_0 a}{2(2\mu+ka)}, \tag{12}$$

with κ being as for plane stress. Again Lamé's solution is recovered when $k=0$, and a state of uniform all-round tension obtains when $a\to\delta/2$ provided k is taken so that it passes the cohesive-law patch test in spherical polar coordinates ($k=8\mu/\delta(3\kappa-5)$).

Implicit in both the circular and spherical hole problems treated here is the existence of a length scale which is considerably larger than the initial radii, and which remains fixed as radii go to zero. This additional length scale can be made explicit by instead considering an annular plate and a hollow ball. The same anomalies result when internal holes are shrunk to zero: They can be remedied by a parallel introduction of cohesive laws.

It is also possible to adapt the foregoing if one actually wanted to model an imperfection. Then the fact that material on opposite sides of the holes had once been separated can be reflected in the choice of the cohesive law as material gets back together if indeed there is some impediment which modifies this law. To be truly physically appropriate, this choice needs to be founded in solid-state physics. Such an analysis is beyond the scope of the present note.

In sum, the boundary conditions in Lamé's classical solutions for elastic solids with holes are not physically appropriate when

[2] If instead r is not fixed in terms of a before taking the limit $a\to 0$, then a state of all-round tension obtains (see (1)). This is a different limit, however, since under it one is moving to infinity rather than to the center of the hole.

[3] Insertion of an entire, nonlinear, cohesive, stress-separation law is tractable within linear elasticity because the present problems are one-dimensional. It is not appropriate, though, because the large strains incurred near the peak stresses in cohesive laws really require a finite strain analysis.

hole surfaces come into extremely close proximity with one another. Cohesive stresses act under these circumstances. Without such proximities though, classical solutions are applicable.

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