Example involving the circular arch

Last class I indicated the steps to solve the problem below. Here I am providing the actual solution so that you can check your answers and make sure that you understand how to solve the problem.

Problem statement:
Find the member end-forces for the fixed-end circular arch of the figure below with a radius of 100ft, arch angle 120°, under a concentrated load of 15 kips acting at the crown of the arch. The moment of inertia of the arch cross-section is I=46,656 in⁴.

Solution. The arch is divided into two elements. The degrees of freedom are shown in Fig. The arch angle β is the same for both elements. The start angle α is 120° for the first element, on the right, and 180° for the second element. The stiffness matrix for the two elements in the local coordinates is

\[
\begin{bmatrix}
U_1 \\
V_1 \\
M_1 \\
U_2 \\
V_2 \\
M_2
\end{bmatrix} = \begin{bmatrix}
-400 & 200 & 40400 & -300 & 200 & -45600 \\
200 & 100 & 29300 & -200 & 100 & -20300 \\
40400 & 29300 & 9487400 & -45600 & 20300 & -3263900 \\
-300 & -200 & -45600 & 400 & -200 & 40400 \\
200 & 100 & 20300 & -200 & 100 & -29300 \\
-35600 & -20300 & -3263900 & 40400 & -29300 & 9487400
\end{bmatrix}
\begin{bmatrix}
u_1 \\
v_1 \\
\theta_1 \\
u_2 \\
v_2 \\
\theta_2
\end{bmatrix}
\]

The transformation matrices for the two elements are

\[
\Gamma_1 = \begin{bmatrix}
-0.5 & 0.866 & 0 & 0 & 0 & 0 \\
-0.866 & -0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1.0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1.0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.0
\end{bmatrix}
\]

PLEASE NOTICE:

\[ [\tau] = [\Gamma]^T \Gamma \]

\[ \Gamma \]

\[ \tau \]
\[
\Gamma_2 = \begin{bmatrix}
-1.0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1.0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5 & -0.866 & 0 \\
0 & 0 & 0 & 0.866 & -0.5 & 0 \\
0 & 0 & 0 & 0 & 1.0 & 0
\end{bmatrix}
\]

The force–displacement relations in the global coordinates are \( \tilde{s} = \tilde{K}s \), with \( \tilde{K} = \Gamma^T K \Gamma \). This results in the force–displacement relations in the global coordinates as follows.

\[
\begin{bmatrix}
\tilde{U}_1 \\
\tilde{V}_1 \\
\tilde{M}_1 \\
\tilde{U}_2 \\
\tilde{V}_2 \\
\tilde{M}_2
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & r_1 & r_2 & r_3 \\
-400 & -200 & -45600 & -400 & 200 & 40400 \\
-200 & 100 & 20300 & 200 & -100 & -29300 \\
-45600 & 20300 & 94874400 & 45600 & -20300 & -3263900 \\
-400 & 200 & 45600 & 400 & -200 & -40400 \\
200 & -100 & -20300 & -200 & 100 & 29300
\end{bmatrix} \begin{bmatrix}
0 \\
r_1 \\
r_2 \\
r_3 \\
r_1 \\
r_2 \\
r_3
\end{bmatrix}
\]

The degrees of freedom associated with the columns of the global stiffness matrices are indicated above the matrices. The structure stiffness matrix is found by assembling the global stiffness matrices in the same way as discussed for continuous beams. This gives the following equilibrium equations:

\[
\begin{bmatrix}
800 & 0 & -80800 \\
0 & 200 & 0 \\
-80800 & 0 & 18974800
\end{bmatrix} \begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix} = \begin{bmatrix}
0 \\
-10 \\
0
\end{bmatrix}
\]

The solution of the above equation gives the displacements,

\[
\begin{bmatrix}
r_1 \\
r_2 \\
r_3
\end{bmatrix} = \begin{bmatrix}
0 \\
-0.075 \\
0
\end{bmatrix}
\]

Thus, the member displacement vectors in the global coordinates are \( \tilde{s} \) and calculated as:

\[
\begin{bmatrix}
U_1 \\
V_1 \\
M_1 \\
U_2 \\
V_2 \\
M_2
\end{bmatrix} = \begin{bmatrix}
828 \\
5 \\
828 \\
-8 \\
-5 \\
-1193
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
8 \\
8 \\
5 \\
1193 \\
-8 \\
-828
\end{bmatrix}
\]