Handout on Method of Consistent Deformations

Chapter 5

Method of Consistent Deformations

5-1 GENERAL

Statically indeterminate structures can be analyzed by direct use of the theory of elastic deformations developed in Chapter 4. Any statically indeterminate structure can be made statically determinate and stable by removing the extra restraints called redundant forces or statical redundants, that is, the force elements that are more than the minimum necessary for the static equilibrium of the structure. The number of redundant forces therefore represents the degrees of statical indeterminacy of the original structure. The statically determinate and stable structure that remains after removal of the extra restraints is called the primary, or released, structure. The choice of the redundant forces is arbitrary. They may be external support reactions or internal member forces or both. In all cases, the statical redundants should be so chosen that the resulting primary structure is stable.

The original structure is then equivalent to the primary structure subjected to the combined action of the original loads plus the unknown redundants. The conditional equations for geometric consistence of the original structure at redundant points (releases), called the compatibility equations, are then obtained from the primary structure by superposition of the deformations caused separately by the original loads and redundants. There can be as many compatibility equations as the number of unknown redundants so that the redundants can be determined by solving these simultaneous equations. This method, known as consistent deformations, is generally applicable to the analysis of any structure, whether it is being analyzed for the effect of loads, support settlement, temperature change, or any other case. However, there is one restriction on the use of this method: the principle of superposition must hold.

As an illustration, consider the loaded continuous beam with nonyielding supports shown in Fig. 5-1(a). It is statically indeterminate to the second degree, that is, with two redundants. The first step in the application of the method is to remove, say, the two interior supports and to introduce in these releases the redundant actions called \( x_1 \) and \( x_2 \), respectively, and by so doing to reduce or cut back the structure to a condition of determinateness and stability. The original structure is now considered as a simple beam (the primary structure) subjected to the combined action of a number of external forces and two redundants \( x_1 \) and \( x_2 \), as shown in Fig. 5-1(b).

The resulting structure in Fig. 5-1(b) can be regarded as the superposition of those shown in Fig. 5-1(c) to (e). Consequently, any deformation of the structure can be obtained by the superposition of these effects.

Referring to Fig. 5-1(b), for unyielding supports we find that compatibility requires

\[
\Delta_1 = 0 \tag{5-1}
\]

\[
\Delta_2 = 0 \tag{5-2}
\]
where $\Delta_1'$ is deflection at redundant point 1 (in the line of redundant force $X_1$)
$\Delta_2'$ is deflection at redundant point 2 (in the line of redundant force $X_2$)

By the principle of superposition, we may expand Eqs. 5-1 and 5-2:

$$\Delta_1' + \Delta_{11} + \Delta_{12} = 0$$ (5-3)
$$\Delta_2' + \Delta_{21} + \Delta_{22} = 0$$ (5-4)

where $\Delta_1'$ is deflection at redundant point 1 due to external loads [see Fig. 5-1(c)]
$\Delta_{11}$ is deflection at redundant point 1 due to redundant force $X_1$ [see Fig. 5-1(d)]
$\Delta_{12}$ is deflection at redundant point 1 due to redundant force $X_2$ [see Fig. 5-1(e)]

The rest are similar.

Equations 5-3 and 5-4 may be expressed in terms of the flexibility coefficients. A typical flexibility coefficient $\delta_{ij}$ is defined by

$$\delta_{ij} = \text{displacement at point } i \text{ due to a unit action at } j, \text{ all other points being assumed unloaded}$$

Thus, Eqs. 5-3 and 5-4 may be written as

$$\Delta_1' + \delta_{11} X_1 + \delta_{12} X_2 = 0$$ (5-5)
$$\Delta_2' + \delta_{21} X_1 + \delta_{22} X_2 = 0$$ (5-6)

Apparently,

$$\delta_{11} = \text{deflection at point 1 due to a unit force at point 1} [\text{see Fig. 5-1(d)}]$$
$$\delta_{12} = \text{deflection at point 1 due to a unit force at point 2} [\text{see Fig. 5-1(e)}]$$

and so on.

Both the deflections resulting from the original external loads and the flexibility coefficients for the primary structure can be obtained by any method described in Chapter 4. The remaining redundant unknowns are then solved by simultaneous equations. In general, for a structure with $n$ redundants, we have

$$\Delta_1' + \delta_{11} X_1 + \delta_{12} X_2 + \cdots + \delta_{1n} X_n = 0$$
$$\Delta_2' + \delta_{21} X_1 + \delta_{22} X_2 + \cdots + \delta_{2n} X_n = 0$$
$$\vdots$$
$$\Delta_n' + \delta_{n1} X_1 + \delta_{n2} X_2 + \cdots + \delta_{nn} X_n = 0$$ (5-7)

In a more general form, we may include the prescribed displacements (other than zeros) occurring at the releases of the original structures. Then these values $\Delta_1, \Delta_2, \ldots$ must be substituted for the zeros on the right-hand side of Eq. 5-8. Thus,

$$\begin{bmatrix} \Delta_1' \\ \Delta_2' \\ \vdots \\ \Delta_n' \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$ (5-8)

or simply

$$\Delta' + F'X = 0$$ (5-9)

in which the column matrix $\Delta'$ on the left-hand side represents the displacements at redundant points of the released structure due to the original loads; the square matrix $F'$ represents the structure flexibility, each column of which gives various deflections at redundant points due to a certain unit redundant force; and the column matrix $X$ on the right-hand side contains the actual displacements at redundant points of the original structure. Equation 5-11 expresses the compatibility at redundant points in terms of unknown redundant forces.

The remainder of this chapter is mostly devoted to the application of Eq. 5-11 to beam, truss, and frame problems using hand computation. At the end, it is shown that Eq. 5-11 can be generated automatically for truss problems through a matrix formulation, and the process is amenable to computer implementation. Further generalization to frame problems is given in Chapter 6.

5-2 ANALYSIS OF STATICALLY INDETERMINATE BEAMS

BY THE METHOD OF CONSISTENT DEFORMATIONS

The method of consistent deformations is easy to understand and can be most effectively demonstrated by a series of illustrations. In all the following examples we assume that only the bending distortion is significant.
where $\Delta_1' = \text{deflection at redundant point 1 (in the line of redundant force } X_1)$
$\Delta_2' = \text{deflection at redundant point 2 (in the line of redundant force } X_2)$

By the principle of superposition, we may expand Eqs. 5-1 and 5-2:

\[
\Delta_1' + \Delta_{11} + \Delta_{12} = 0 \\
\Delta_2' + \Delta_{21} + \Delta_{22} = 0
\]

(5-3)

(5-4)

where $\Delta_1' = \text{deflection at redundant point 1 due to external loads [see Fig. 5-1(c)]}$
$\Delta_{11} = \text{deflection at redundant point 1 due to redundant force } X_1$ [see Fig. 5-1(d)]
$\Delta_{12} = \text{deflection at redundant point 1 due to redundant force } X_2$ [see Fig. 5-1(e)]

The rest are similar.

Equations 5-3 and 5-4 may be expressed in terms of the flexibility coefficients. A typical flexibility coefficient $\delta_{ij}$ is defined by

$\delta_{ij} = \text{displacement at point } i \text{ due to a unit action at } j$, all other points being assumed unloaded

Thus, Eqs. 5-3 and 5-4 may be written as

\[
\Delta_1' + \delta_{11} X_1 + \delta_{12} X_2 + \ldots + \delta_{1n} X_n = 0 \\
\Delta_2' + \delta_{21} X_1 + \delta_{22} X_2 + \ldots + \delta_{2n} X_n = 0
\]

(5-5)

(5-6)

Apparently,

$\delta_{11} = \text{deflection at point 1 due to a unit force at point 1 [see Fig. 5-1(d)]}$
$\delta_{12} = \text{deflection at point 1 due to a unit force at point 2 [see Fig. 5-1(e)]}$

and so on.

Both the deflections resulting from the original external loads and the flexibility coefficients for the primary structure can be obtained by any method described in Chapter 4. The remaining redundant unknowns are then solved by simultaneous equations. In general, for a structure with $n$ redundants, we have

\[
\Delta_1' + \delta_{11} X_1 + \delta_{12} X_2 + \ldots + \delta_{1n} X_n = 0 \\
\Delta_2' + \delta_{21} X_1 + \delta_{22} X_2 + \ldots + \delta_{2n} X_n = 0 \\
\vdots \\
\Delta_n' + \delta_{n1} X_1 + \delta_{n2} X_2 + \ldots + \delta_{nn} X_n = 0
\]

(5-7)

Equation 5-7 in matrix form is

\[
\begin{bmatrix}
\Delta_1' \\
\Delta_2' \\
\vdots \\
\Delta_n'
\end{bmatrix} +
\begin{bmatrix}
\delta_{11} & \delta_{12} & \ldots & \delta_{1n} \\
\delta_{21} & \delta_{22} & \ldots & \delta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{n1} & \delta_{n2} & \ldots & \delta_{nn}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix} =
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\vdots \\
\Delta_n
\end{bmatrix}
\]

(5-8)

or simply

\[
\Delta' + F'X = 0
\]

(5-9)

In a more general form, we may include the prescribed displacements (other than zeros) occurring at the releases of the original structures. Then these values $\Delta_1, \Delta_2, \ldots$ must be substituted for the zeros on the right-hand side of Eq. 5-8. Thus,

\[
\begin{bmatrix}
\Delta_1' \\
\Delta_2' \\
\vdots \\
\Delta_n'
\end{bmatrix} +
\begin{bmatrix}
\delta_{11} & \delta_{12} & \ldots & \delta_{1n} \\
\delta_{21} & \delta_{22} & \ldots & \delta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{n1} & \delta_{n2} & \ldots & \delta_{nn}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix} =
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\vdots \\
\Delta_n
\end{bmatrix}
\]

(5-10)

or simply

\[
\Delta' + F'X = \Delta
\]

(5-11)

in which the column matrix $\Delta'$ on the left-hand side represents the displacements at redundant points of the released structure due to the original loads; the square matrix $F'$ represents the structure flexibility, each column of which gives various displacements at redundant points due to a certain unit redundant force; and the column matrix $\Delta$ on the right-hand side contains the actual displacements at redundant points of the original structure. Equation 5-11 expresses the compatibility at redundant points in terms of unknown redundant forces.

The remainder of this chapter is mostly devoted to the application of Eq. 5-11 to beam, truss, and frame problems using hand computation. At the end, it is shown that Eq. 5-11 can be generated automatically for truss problems through a matrix formulation, and the process is amenable to computer implementation. Further generalization to frame problems is given in Chapter 6.

### 5-2 ANALYSIS OF STATICALLY INDETERMINATE BEAMS

BY THE METHOD OF CONSISTENT DEFORMATIONS

The method of consistent deformations is easy to understand and can be most effectively demonstrated by a series of illustrations. In all the following examples we assume that only the bending distortion is significant.
Example 5-1

Analyze the propped beam shown in Fig. 5-2(a), which is statically indeterminate to the first degree. Assume constant $EI$.

**Solution 1**  One reaction may be considered as being extra. In this case let us first choose the vertical reaction at $b$ as the redundant assumed to be acting downward, as shown in Fig. 5-2(b). By the principle of superposition, we may consider the beam as being subjected to the sum of the effects of the original uniform loading and the unknown redundant $X_b$, as shown in Fig. 5-2(c) and (d), respectively.

Next, we find that the vertical deflection at $b$ resulting from the uniform load [Fig. 5-2(c)] is given by

$$\Delta_b = \frac{wL^4}{8EI}$$

and that the vertical deflection at $b$ because of a unit load applied at $b$ in place of $X_b$ [Fig. 5-2(d)] is given by

$$\Delta_b = \frac{wL^4}{8EI} + \left(\frac{1^3}{3EI}\right)X_b = 0$$

from which

$$X_b = -\frac{3wL}{8}$$

The minus sign indicates an upward reaction.

With reaction at $b$ determined, we find that the beam reduces to a statically determinate one. We can readily obtain reaction components at $a$ from the equilibrium equations:

$$\sum F_x = 0, \quad V_a = \frac{wL}{2} \quad \text{w} \quad \text{w} = \frac{wL}{2} \quad \text{w} \quad \text{w}$$

(counterclockwise)

The moment diagram for the beam is shown in Fig. 5-2(e).

**Solution 2**  The beam in Fig. 5-2(a) can be rendered statically determinate by removing the fixed support and replacing it with a hinged support. In addition to the original uniform loading, a redundant moment $M_a$ is then applied to the primary structure, a simple beam, as shown in Fig. 5-3(a). The unknown $M_a$ can be solved by the condition of compatibility that the rotation at end $a$ must be zero.

The rotation at end $a$ for the primary structure due to the uniform loading alone [Fig. 5-3(b)] is given by

$$\theta_a = \frac{wL^3}{24EI}$$

and that due to a unit couple applied at end $a$ [Fig. 5-3(c)] is given by

$$\theta_a = \theta_a' + \theta_{ac} = \frac{wL^3}{24EI} + \frac{M_aL}{3EI} = 0$$

Using the compatibility equation

$$\theta_a = \theta_a' + \theta_{ac} = \frac{wL^3}{24EI} + \frac{M_aL}{3EI} = 0$$
Method of Consistent Deformations  Chapter 5

![Figure 5-3](image)

we solve for

\[ M_a = -\frac{1}{2}wl^2 \]

The minus sign indicates a counterclockwise moment. After \( M_a \) is determined, the rest of the analysis can be carried out without difficulty.

**Solution 3** From the previous solutions we recognize that we are free to select redundant in analyzing a statically indeterminate structure, the only restriction being that the redundant should be so selected that a stable cut structure remains. Figure 5-4 will serve as an illustration. Let us cut the beam at midspan section \( c \) and introduce in its place a hinge so that the beam is stable and determinate. A pair of redundant couples, called \( M_a \), together with the original loading are then applied to the primary structure, as shown in Fig. 5-4(a).

The redundant \( M_a \) is solved by the condition of compatibility that the rotation of the left side relative to the right side at section \( c \) must be zero.

Using the method of virtual force, we evaluate the relative rotation at \( c \) due to the external loading alone [Fig. 5-4(b)] as

\[ \theta'_c = \int_0^l \frac{Mm}{EI} dx = \int_0^l \left[ \frac{(wx/4) - (wx^2/2l)}{EI} \right] 2x/3l \, dx = -\frac{wl^3}{12EI} \]

and that due to a pair of unit couples acting at \( c \) [Fig. 5-4(c)] as

\[ \delta_{cw} = \int_0^l m^2 x dx = \int_0^l \left( \frac{2x^2}{l^2} \right) 2x/3l \, dx = \frac{4l}{3EI} \]

Setting the total relative angular displacement at \( c \) equal to zero, we have

\[ -\frac{wl^3}{12EI} + M_a \left( \frac{4l}{3EI} \right) = 0 \]

from which

\[ M_a = \frac{wl^2}{16} \]

After \( M_a \) is determined, the rest of the analysis can easily be carried out.

**Example 5-2**

Suppose that the support at \( b \) of Example 5-1 is elastic and the spring flexibility is \( f \) (displacement per unit force), as shown in Fig. 5-5. Determine the reaction at \( b \) (the spring force), denoted by \( X_b \).

Assume downward \( X_b \) (i.e., tension in the spring) as positive. The compatibility is

\[ \Delta_c + \delta_{cb} X_b + fX_b = 0 \]

This equation can be explained by putting it in the form

\[ \Delta_c - \delta_{cb} (-X_b) = f(-X_b) \]

Since \(-X_b\) represents the compression in the spring, the equation indicates that the downward deflection at \( b \) caused by the beam load minus that caused by upward reaction should be equal to the spring contraction.

By substituting \( \Delta_c = \frac{wl^3}{8EI}, \delta_{cb} = \frac{l^3}{3EI} \) in the preceding equation, we obtain

\[ \frac{wl^3}{8EI} + \frac{X_b l^3}{3EI} + fX_b = 0 \]
Method of Consistent Deformations  Chapter 5

from which

\[ X_b = -\frac{3}{8} w l \left[ \frac{1}{1 + (3fEI/l^2)} \right] \]

The minus sign indicates an upward reaction.

For a nonyielding support, \( f = 0 \), the preceding equation gives

\[ X_b = -\frac{1}{4} w l \]

as found previously.

If a beam is provided with \( n \) redundant elastic supports having spring flexibilities \( f_1, f_2, \ldots, f_n \), respectively, then the general compatibility equation is

\[
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\vdots \\
\Delta_n
\end{bmatrix} +
\begin{bmatrix}
\delta_{11} + f_1 & \delta_{12} & \cdots & \delta_{1n} \\
\delta_{21} & \delta_{22} + f_2 & \cdots & \delta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} + f_n
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix} = 0
\]

(5-12)

Example 5-3

Find the reactions for the beam with two sections shown in Fig. 5-6(a).

In this problem it may be convenient to select the vertical reaction at support \( b \) as redundant. The beam is then considered as a simple beam subject to the original loading and the redundant \( R_b \), as shown in Fig. 5-6(b) and (c), respectively.

The compatibility requires

\[ \Delta_b = \Delta_0 + \delta_{nn} R_b = 0 \]

Using the method of virtual force, we have

\[ \int \frac{M m_b}{EI} \, dx + R_b \int \frac{(m_b)^2 \, dx}{EI} = 0 \]

from which

\[ R_b = -\frac{\int M m_b \, dx}{\int (m_b)^2 \, dx} \]

Section 5-2 Analysis of Statically Indeterminate Beams

The solution is completely shown in Table 5-1.

where \( M = \) bending moment at any section of the primary beam caused by the original loading [Fig. 5-6(a)]

\( m_b = \) bending moment at the same section of the primary beam caused by a unit load in place of the redundant \( R_b \) [Fig. 5-6(c)]

\( R_b = -\frac{\int_0^L \frac{20x - x^2}{2} \, dx}{\int_0^L \frac{200(x/2)^2}{2E I} \, dx} - \frac{\int_0^L \frac{200(x/2)^2}{2E I} \, dx}{\int_0^L \frac{200(x/2)^2}{2E I} \, dx} = -25.84 \text{ kips} \)

The negative sign indicates an upward reaction at support \( b \).

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<th>TABLE 5-1</th>
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from which

\[ X_b = -\frac{3}{8}wl \left[ \frac{1}{1 + (3FEI/l^3)} \right] \]

The minus sign indicates an upward reaction.

For a non-yielding support, \( f = 0 \), the preceding equation gives

\[ X_b = -\frac{3}{8}wl \]

as found previously.

If a beam is provided with \( n \) redundant elastic supports having spring flexibilities \( f_1, f_2, \ldots, f_n \), respectively, then the general compatibility equation is

\[
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\vdots \\
\Delta_n
\end{bmatrix} + \begin{bmatrix}
\delta_{11} + f_1 & \delta_{12} & \cdots & \delta_{1n} \\
\delta_{21} & \delta_{22} + f_2 & \cdots & \delta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} + f_n
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix} \quad (5-12)
\]

Example 5-3

Find the reactions for the beam with two sections shown in Fig. 5-6(a).

In this problem it may be convenient to select the vertical reaction at support \( b \) as redundant. The beam is then considered as a simple beam subject to the original loading and the redundant \( R_b \), as shown in Fig. 5-6(b) and (c), respectively.

The compatibility requires

\[ \Delta_b = \Delta'_b + \delta_{bb} R_b = 0 \]

Using the method of virtual force, we have

\[ \int \frac{M_m dx}{EI} + R_b \left( \int \frac{m_b x^2 dx}{EI} \right) = 0 \]

from which

\[ R_b = -\frac{\int M_m x dx /EI}{\int m_b x^2 dx /EI} \]

where \( M = \) bending moment at any section of the primary beam caused by the original loading [Fig. 5-6(b)]

\( m_b = \) bending moment at the same section of the primary beam caused by a unit load in place of the redundant \( R_b \) [Fig. 5-6(c)]

The solution is completely shown in Table 5-1.

\[
R_b = -\left[ \int_0^{10} \frac{(20x-x^2/2)x/2 dx}{2EI} + \int_0^{10} \frac{(20x-x^2/2)x/2 dx}{2EI} + \int_0^{20} \frac{(20x)^2/2 dx}{2EI} \right]
\]

\[ = -25.84 \text{ kips} \]

The negative sign indicates an upward reaction at support \( b \).

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After $R_s$ is obtained, we can readily find the reactions at the other two supports by statics. That is,

$$R_s = R_d = 20 - (\frac{1}{2})(25.84) = 7.08 \text{ kips}$$

acting upward.

The end moments for a fixed-end beam, called fixed-end moments, are important in the methods of slope-deflection and of moment distribution, which are discussed in later chapters. The following examples are attempts to solve fixed-end moments due to common types of loading by the method of consistent deformations.

Example 5-4

The fixed-end beam of uniform cross section subjected to a single concentrated load shown in Fig. 5-7(a) is statically indeterminate to the second degree since the horizontal force does not exist. End moments $M_A$ and $M_B$ are selected as redundants. The original beam is then considered as equivalent to a simple beam (not shown) under the combined action of a concentrated force $P$ and redundant moments $M_A$ and $M_B$. It is convenient to apply the conjugate beam method to determine $M_A$ and $M_B$ based on the condition that the slope and deflection at either end of the fixed-end beam must be zero. In other words, there will be no support reactions for the conjugate beam, and the positive and negative $M/EI$ diagrams (elastic loads) given in Fig. 5-7(b) must form a balanced system. Thus, from $\Sigma F_y = 0$,

$$\frac{Pab}{2EI} - \frac{M_A l}{2EI} - \frac{M_B l}{2EI} = 0$$

or

$$M_A + M_B = \frac{Pab}{l} \quad (5-13)$$

From $\Sigma M_B = 0$,

$$\left(\frac{Pab}{2EI}\left(\frac{l + b}{3}\right) - \frac{M_A l}{2EI}\left(\frac{2l}{3}\right) - \frac{M_B l}{2EI}\left(\frac{l}{3}\right)\right) = 0$$

or

$$2M_A + M_B = \frac{Pab}{l} + \frac{Pab^2}{l^2} \quad (5-14)$$

Solving Eqs. 5-13 and 5-14 simultaneously, we obtain

$$M_A = \frac{Pab^2}{l^2}, \quad M_B = \frac{Pab^2}{l^2} \quad (5-15)$$

Example 5-5

Find the end moments of a fixed-end beam of constant $EI$ caused by a uniform load, as shown in Fig. 5-8(a).

Because of symmetry, the beam is statically indeterminate to the first degree, since $M_A = M_B = M$, as indicated in Fig. 5-8(a). By the method of conjugate beam [Fig. 5-8(b)], $\Sigma F_y = 0$,

$$\left(\frac{w l^2}{8EI}\left(\frac{2l}{3}\right) - \frac{M l}{EI}\right) = 0$$

from which

$$M = \frac{1}{12} w l^2 \quad (5-16)$$

![Figure 5-7](image)

![Figure 5-8](image)
Example 5-6

If the fixed-end beam is loaded with an external couple \( M \) as shown in Fig. 5-9(a), the deflected elastic shape will be somewhat like that shown by the dashed line, which gives the sense of the end moments as indicated.

As before, end moments \( M_A \) and \( M_B \) are chosen as redundants. The elastic loads based on the moment diagrams divided by \( EI \) plotted for external moment \( M \) and redundants \( M_A \) and \( M_B \), as given in Fig. 5-9(b) and (c), must be in equilibrium themselves. From \( \Sigma F_y = 0 \),

\[
\frac{M_A l}{2EI} + \frac{M b^2}{2EI} - \frac{M a^2}{2EI} = 0
\]

or

\[
M_A - M_B = \frac{M(a^2 - b^2)}{l^2}
\]

(5-17)

From \( \Sigma M_B = 0 \),

\[
\left( \frac{M_A l}{2EI} \right) \left( \frac{2l}{3} \right) + \left( \frac{M b^2}{2EI} \right) \left( \frac{2b}{3} \right) - \left( \frac{M a^2}{2EI} \right) \left( \frac{a}{3} \right) = 0
\]

(5-18)

Solving Eqs. 5-17 and 5-18 simultaneously, we obtain

\[
M_A = \frac{M b}{l^2} (2a - b), \quad M_B = \frac{M a}{l^2} (2b - a)
\]

(5-19)

Note that \( M_A \) and \( M_B \) bear the same sense as the externally applied \( M \), as indicated in Fig. 5-9(a), if \( a > l/3 \) and \( b > l/3 \).

5-3 ANALYSIS OF STATICALLY INDETERMINATE RIGID FRAMES

BY THE METHOD OF CONSISTENT DEFORMATIONS

The general procedure illustrated in Sec. 5-2 in solving statically indeterminate beams can be applied equally well to the analysis of statically indeterminate rigid frames, as in the following example.

Example 5-7

For the loaded rigid frame shown in Fig. 5-10(a), find the reaction components at the fixed end \( a \), and plot the moment diagram for the entire frame. Assume the same \( EI \) for all members.

To do this, we start by removing support \( a \) and introducing in its place three redundant reaction components \( X_1 \), \( X_2 \), and \( X_3 \), as shown in Fig. 5-10(b). These can be taken as the superposition of four basic cases, as shown in Fig. 5-10(c), (d), (e), and (f), respectively. Since end \( a \) is fixed, compatibility requires that

\[
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\Delta_3
\end{bmatrix} +
\begin{bmatrix}
\delta_{11} & \delta_{12} & \delta_{13} \\
\delta_{21} & \delta_{22} & \delta_{23} \\
\delta_{31} & \delta_{32} & \delta_{33}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(5-20)

Taking advantage of Examples 4-7 and 4-8, we note that

\[
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\Delta_3
\end{bmatrix} = \frac{1}{EI}
\begin{bmatrix}
5,000 \\
-7,500 \\
-800
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\delta_{11} & \delta_{12} & \delta_{13} \\
\delta_{21} & \delta_{22} & \delta_{23} \\
\delta_{31} & \delta_{32} & \delta_{33}
\end{bmatrix} = \frac{1}{EI}
\begin{bmatrix}
1,667 & -1,000 & -200 \\
-1,000 & 1,333 & 150 \\
-200 & 150 & 30
\end{bmatrix}
\]
Solving, we obtain
\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} = \begin{bmatrix}
1 \\
6 \\
3.33 ft
\end{bmatrix} \text{ kips}
\]

Note that the solution of this problem could be simplified by setting \(X_2 = 6\) kips in Eq. 5-20, since we know this value beforehand because of the symmetry of the loaded frame.

The final results are shown in Fig. 5-10(g); the moment diagram for the whole frame is shown in Fig. 5-10(h). A sketch of the elastic deformation of the frame due to bending distortion is shown by the dashed line in Fig. 5-10(i). Note that in this case there is one point of inflection in each column and two points of inflection in the beam.

By referring to Example 5-7, we see that by using the method of consistent deformations in analyzing a rigid frame, we encounter tedious calculations of the flexibility coefficients. The work, if done by hand, will become intolerable if the problem involves as many redundants as a rigid frame usually does. As a matter of fact, the method of consistent deformations is seldom used for analysis of rigid frames by hand calculation, since a solution can be much more easily obtained by the method of slope-deflection or of moment distribution. However, with the development of high-speed electronic computers, this method can be made amenable to computers through a matrix formulation, as shown in Sec. 5-6 and Chapter 6.

5-4 ANALYSIS OF STATICALLY INDETERMINATE TRUSSES BY THE METHOD OF CONSISTENT DEFORMATIONS

The indeterminateness of a truss may be due to redundant supports or redundant bars or both. If it results from redundant supports, the procedure for attack is the same as that described for a continuous beam. If the superfluous element is a bar, the bar is considered to be cut at a section and replaced by two equal and opposite axial redundant forces representing the internal action for that bar. The condition equation is such that the relative axial displacement between the two sides at the cut section caused by the combined effect of the original loading and the redundants should be zero.

Example 5-8

Analyze the continuous truss in Fig. 5-11(a). Assume that \(E = 30,000\ \text{kips/in.}^2\) and \(L\ (\text{ft})/4 \ (\text{in.}^2) = 1\) for all members.

In this problem it is convenient to select the central support as the redundant element. We begin by removing support \(c\) and introducing in its place a redundant reaction \(X_r\), as shown in Fig. 5-11(b). The primary structure is then a simply supported truss subjected to an external load of 64 kips at joint \(b\) and a redundant \(X_r\). The effects can be separated, respectively, as shown in Fig. 5-11(c) and (d).

Since support \(c\) is on a rigid foundation, the compatibility equation can be expressed by
\[
\Delta_c = \Delta'_c + 5_c X_r = 0
\]
Using virtual force gives
\[ \sum S'u_i L/AE + X_c \sum u_i^2 L/AE = 0 \]

from which
\[ X_c = -\frac{\sum (S'u_i L/AE)}{\sum (u_i^2 L/AE)} \]

where \( S' \) = internal force in any member of the primary truss due to the original loading [Fig. 5-11(c)]
\( u_c \) = internal force in the same member of the primary truss due to a unit force at \( c \) [Fig. 5-11(d)]

The solution is shown completely in Table 5-2.

The negative sign indicates an upward reaction at support \( c \).

After \( X_c \) is determined, we can readily obtain each bar force \( S \) from
\[ S = S' + u_c X_c \]

as given in the last column of Table 5-2.

Example 5-9
Analyze the truss in Fig. 5-12(a). Assume that \( E = 30,000 \text{ kips/in.}^2 \) and \( L \text{ (ft)/} A \text{ (in.}^2\) = 1 for all members.
Method of Consistent Deformations

Chapter 5

The truss in Fig. 5-12(a) has two redundant elements, one in the reaction component and the other in the bar. Let us select the horizontal component of reaction at the right-end hinge and the internal force in bar Cd as redundants. We then have a primary truss loaded, as shown in Fig. 5-12(b), in which the original hinged support e is replaced by a roller acted on by a redundant horizontal reaction \(X_1\), and the bar Cd is cut and a pair of redundant forces \(X_2\) applied to it. This may again be replaced by the three basic cases shown in Fig. 5-12(c) to (e). Since both the horizontal movement at support e and the relative axial displacement between the cut ends of bar Cd are zero, we have

\[
\begin{align*}
\left[ \Delta_1 \right] &= \left[ \Delta_2 \right] + \left[ \delta_{11} \delta_{12} \delta_{21} \delta_{22} \right] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (5-22) \\
\sum S' u_1 L &+ \sum \frac{u_2 L}{AE} + \sum S' u_3 L &+ \sum \frac{u_2 L}{AE} \\
\sum S' u_1 L &+ \sum \frac{u_2 L}{AE} &+ \sum S' u_2 L &+ \sum \frac{u_2 L}{AE} \\
\sum S' u_1 L &+ \sum \frac{u_2 L}{AE} &+ \sum S' u_2 L &+ \sum \frac{u_2 L}{AE} \\
\sum S' u_1 L &+ \sum \frac{u_2 L}{AE} &+ \sum S' u_2 L &+ \sum \frac{u_2 L}{AE} \\
S' &= X_1 + X_2, \\
\sum S' = X_1 + X_2, \\
S' &= X_1 + X_2, \ \\
\sum S' = X_1 + X_2.
\end{align*}
\]

or

Note that

\( S' = \) internal force in any bar of the primary truss due to the original loading [Fig. 5-12(c)]

\( u_1 = \) internal force in the same bar of the primary truss due to a unit horizontal force acting at e [Fig. 5-12(d)]

\( u_2 = \) internal force in the same bar of the primary truss due to a pair of unit axial forces acting at the cut ends of bar Cd [Fig. 5-12(e)]

Using the values summed up in Table 5-3, we reduce Eq. 5-23 to

\[
\begin{align*}
\begin{bmatrix} 96 \\ 27.2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
(5-24)
\end{align*}
\]

Solving, we obtain

\[
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -25.6 \\ -10.6 \end{bmatrix} \text{ kips}
\]

The negative signs indicate that the horizontal reaction at hinge e acts to the left and that the axial force in member Cd is compressive. The rest of the member forces are obtained by

\[
\begin{align*}
S' &= X_1 + X_2, \\
S' &= X_1 + X_2, \\
S' &= X_1 + X_2. 
\end{align*}
\]

The complete solution is shown in Table 5-3.

Example 5-10

Analyze the truss in Fig. 5-13(a) subject to a rise of 50°F at the top chords BC and CD. Assume \( \alpha = 0.0000065 \) in./in.°F, \( E = 30,000 \) kips/in.², and \( L \) (ft)/A (in.)² = 1 for all members.

The truss is statically indeterminate to the first degree. Cut bar Cd and select its
bar force $X_1$ as the redundant as shown in Fig. 5-13(b). The primary truss is then a simply supported truss subjected to the temperature rise in the top chord and the redundant axial force $X_1$. Since the relative axial displacement between the cut ends due to the combined effect of temperature rise and $X_1$ must be zero, we have

$$\Delta = \Delta_1 + \delta_{11} X_1 = 0$$

or

$$\sum u_i (a^o L) + X_1 \sum \frac{u_i L}{AE} = 0$$

where $\Delta_1$ = relative displacement between the cut ends of the primary truss due to the temperature rise = $\Sigma u_i (a^o L)$ (see Eq. 4-26)

$u_i$ = internal force in any member of the primary truss due to a pair of unit axial forces acting at the ends of the cut section.

The solution is shown in Table 5-4.

<table>
<thead>
<tr>
<th>Member</th>
<th>$L$ (ft/in.$^2$)</th>
<th>$u_i$ (ft)</th>
<th>$a_i L$</th>
<th>$u_i a_i L$ (ft)$^2$</th>
<th>$\frac{u_i L}{AE}$ (ft/in.$^2$)</th>
<th>$S = u_i X_1$ (kips)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ab</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>bc</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>cd</td>
<td>1</td>
<td>$-\frac{3}{8}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{3}{8}$</td>
<td>$-21.1$</td>
</tr>
<tr>
<td>de</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>BC</td>
<td>1</td>
<td>0</td>
<td>+0.0078</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CD</td>
<td>1</td>
<td>$-\frac{3}{8}$</td>
<td>+0.0078</td>
<td>$-0.00468$</td>
<td>$\frac{3}{8}$</td>
<td>$-21.1$</td>
</tr>
<tr>
<td>aB</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>bB</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Bc</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Cc</td>
<td>1</td>
<td>$-\frac{3}{8}$</td>
<td>0</td>
<td>$+\frac{3}{8}$</td>
<td>0</td>
<td>$-28.1$</td>
</tr>
<tr>
<td>Cd</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>dD</td>
<td>1</td>
<td>$-\frac{3}{8}$</td>
<td>0</td>
<td>$+\frac{3}{8}$</td>
<td>0</td>
<td>$-28.1$</td>
</tr>
<tr>
<td>De</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td></td>
<td></td>
<td></td>
<td>$-0.00468$</td>
<td>+4</td>
<td></td>
</tr>
</tbody>
</table>

Section 5-5  Castigliano's Compatibility Equations (Method of Least Work)

$$\frac{4X_1}{30,000} = 0.00468 = 0$$

$$X_1 = 35.1 \text{kips (tension)}$$

Although these illustrations are aimed at statically indeterminate trusses with one or two redundants, the procedure described can be extended to trusses with many degrees of redundancy.

5-5 CASTIGLIANO'S COMPATIBILITY EQUATION
(METHOD OF LEAST WORK)

The method of consistent deformations hitherto discussed involves superposition equations for the elastic deformations of the primary structure at the points of application of the redundants $X_1, X_2, \ldots, X_n$, the primary structure being stable and determinate and subjected to external actions, together with $n$ redundant forces. The expression that the displacement at each of $n$ redundants equals zero for a loaded structure with nonyielding supports may be set up by the use of Castigliano's theorem as

$$\begin{bmatrix} \frac{\partial W}{\partial X_1} \\ \frac{\partial W}{\partial X_2} \\ \vdots \\ \frac{\partial W}{\partial X_n} \end{bmatrix} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} = 0$$

(5-25)

where $W$ is the total strain energy of the primary structure and is therefore a function of the external loads and the unknown redundant forces $X_1, X_2, \ldots, X_n$. There are as many simultaneous equations as the number of unknown redundants involved. Equation 5-25,

$$\frac{\partial W}{\partial X_1} = \frac{\partial W}{\partial X_2} = \cdots = \frac{\partial W}{\partial X_n} = 0$$

is known as Castigliano's compatibility equation and it may be stated as follows: The redundants must have such value that the total strain energy of the structure is a minimum consistent with equilibrium. For this reason it is sometimes referred to as the theorem of least work. Note that Castigliano's compatibility equation is limited to the computation of redundant forces produced only by external loads on a structure mounted on nonyielding supports. It cannot be used to determine stresses caused by temperature change, support movements, fabrication errors, and the like.

In the analysis of statically indeterminate beams or rigid frames, we consider bending moment to be the only significant factor contributing to the internal energy.
Therefore, the total strain energy can be expressed by

$$ W = \int \frac{M^2}{2EI} dx $$

Setting the derivative of this expression with respect to any redundant $X_i$ equal to zero gives

$$ \int \frac{M(\partial M/\partial X_i)}{EI} dx = 0 $$

Therefore, for a statically indeterminate beam (or rigid frame) with $n$ redundants, we can write a set of $n$ simultaneous compatibility equations:

$$ \begin{bmatrix} \frac{\partial W}{\partial X_1} \\ \frac{\partial W}{\partial X_2} \\ \vdots \\ \frac{\partial W}{\partial X_n} \end{bmatrix} \begin{bmatrix} \int \frac{M(\partial M/\partial X_1)}{EI} dx \\ \int \frac{M(\partial M/\partial X_2)}{EI} dx \\ \vdots \\ \int \frac{M(\partial M/\partial X_n)}{EI} dx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} $$

(5-26)

to solve all the unknown redundants.

In the analysis of statically indeterminate trusses, the total strain energy can be expressed by

$$ W = \sum \frac{S^2L}{2AE} $$

Setting the derivative of this expression with respect to any redundant $X_i$ equal to zero gives

$$ \sum \frac{S(\partial S/\partial X_i)L}{AE} = 0 $$

Thus, for a statically indeterminate truss with $n$ redundant elements, we have a set of $n$ simultaneous compatibility equations available for their solution:

$$ \begin{bmatrix} \sum \frac{S(\partial S/\partial X_1)L}{AE} \\ \sum \frac{S(\partial S/\partial X_2)L}{AE} \\ \vdots \\ \sum \frac{S(\partial S/\partial X_n)L}{AE} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} $$

(5-27)

**Example 5-11**

For the fixed-end beam under general loading shown in Fig. 5-14(a), derive a working formula for solving the end reactions at $A$.

We select the left-end reaction components $M_A$ and $V_A$ as redundants, as shown in Fig. 5-14(b). The primary structure is a cantilever subjected to the original loads on the span together with the redundant forces $M_A$ and $V_A$ at the left end. Applying the method of least work, we obtain

$$ \frac{\partial W}{\partial M_A} = \int_0^l \frac{M(\partial M/\partial M_A)}{EI} dx = 0 $$

(5-28)

$$ \frac{\partial W}{\partial V_A} = \int_0^l \frac{M(\partial M/\partial V_A)}{EI} dx = 0 $$

(5-29)

Since the bending moment at any section of the primary structure is given by

$$ M = M' + M_A + V_Ax $$

where $M'$ indicates the bending moment at the same section of the primary structure resulting from the original loads on the span, we have

$$ \frac{\partial M}{\partial M_A} = 1 \quad \text{and} \quad \frac{\partial M}{\partial V_A} = x $$

Substituting these in Eqs. 5-28 and 5-29 results in the following two equations:

$$ \int_0^l \frac{M}{EI} dx = 0 $$

(5-30)

$$ \int_0^l \frac{Mx}{EI} dx = 0 $$

(5-31)

to solve for redundants $M_A$ and $V_A$.

For a beam of uniform section with constant $EI$, Eqs. 5-30 and 5-31 reduce to

$$ \int_0^l M dx = 0 $$

(5-32)

![Figure 5-14](image-url)
\[ \int_0^a M \, dx = 0 \quad (5-33) \]

As an illustration, let us find the fixed-end moments of the beam shown in Fig. 5-15. Taking the origin at \( A \), we note that:

\[ M = M_A + V_A x, \quad 0 \leq x \leq a \]

\[ M = M_A + V_A x - P(x - a), \quad a \leq x \leq l \]

Applying Eqs. 5-32 and 5-33 gives:

\[ \int_0^a (M_A + V_A x) \, dx + \int_a^l (M_A + V_A x - P(x - a)) \, dx = 0 \]

or

\[ M_A l + \frac{V_A l^2}{2} - \frac{P b^2}{2} = 0 \quad (5-34) \]

and

\[ \int_0^a (M_A + V_A x) \, dx + \int_a^l (M_A + V_A x - P(x - a)) \, dx = 0 \]

\[ \frac{M_A l^2}{2} + \frac{V_A l^3}{3} - \frac{P b^2(a + 2l)}{6} = 0 \quad (5-35) \]

Solving Eqs. 5-34 and 5-35 simultaneously, we obtain:

\[ M_A = -\frac{P ab^2}{l^2}, \quad V_A = \frac{P b^2(l + 2a)}{l^3} \]

Similarly,

\[ M_b = -\frac{Pa^2 b}{l^2}, \quad V_b = \frac{Pa^2(l + 2b)}{l^3} \]

**Example 5.12**

Analyze the frame shown in Fig. 5-16(a) by taking the internal shear, thrust, and moment in the midspan section of the beam as redundants.

Because of symmetry, the shear must be zero in the midspan section \( e \) of the beam, and only thrust and bending moment are left as redundants, as shown in Fig. 5-16(a). The solution can be simplified by considering only half of the frame, as shown in Fig. 5-16(b) and Table 5-5.

Applying

\[ \frac{\partial W}{\partial M_e} = 0 \quad \text{or} \quad \int_e M(\partial M / \partial M_e) \, dx = 0 \]

we have

\[ \int_0^a \left( M_e - \frac{(1.2)x^2}{2} \right) \, dx = 0 \quad \int_0^a (M_e + H_e x - 15) \, dx = 0 \]

or

\[ 3M_e + 10H_e - 35 = 0 \quad (5-36) \]

**Figure 5-16**

<table>
<thead>
<tr>
<th>Member</th>
<th>Origin</th>
<th>Limit (ft)</th>
<th>( M ) (ft-kips)</th>
<th>( \frac{\partial M}{\partial M_e} )</th>
<th>( \frac{\partial M}{\partial H_e} ) (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( eb )</td>
<td>( e )</td>
<td>0 to 5</td>
<td>( M_e - \frac{(1.2)x^2}{2} )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( ba )</td>
<td>( b )</td>
<td>0 to 10</td>
<td>( M_e + H_e x - 15 )</td>
<td>1</td>
<td>( x )</td>
</tr>
</tbody>
</table>
Applying 
\[ \frac{\partial W}{\partial H_e} = 0 \quad \text{or} \quad \frac{\int M(\partial M/\partial H_e) \, dx}{EI} = 0 \]
we have
\[ \frac{2}{EI} \int_0^1 (M_e + H_e x - 15)x \, dx = 0 \]
or
\[ 3M_e + 20H_e - 45 = 0 \quad (5.37) \]

Solving Eqs. 5.36 and 5.37 simultaneously gives
\[ H_e = 1.0 \text{ kip}, \quad M_e = 8.33 \text{ ft-kips} \]
from which we obtain
\[ H_e = 1.0 \text{ kip}, \quad M_e = 3.33 \text{ ft-kips} \]
as previously found.

For a highly indeterminate rigid frame, such as the one shown in Fig. 5-17(a), the procedure of the analysis remains the same. The frame is statically indeterminate to the 24th degree. We may cut it back to three determinate structures and substitute the redundants \( X_1, X_2, \ldots, X_{24} \) at the cut sections as shown in Fig. 5-17(b).

Section 5-5 Castigliano’s Compatibility Equations (Method of Least Work) 149

From least work, we have 24 equations to solve for all the redundants simultaneously:

\[ \begin{bmatrix} \frac{\partial W}{\partial X_1} & \frac{\partial W}{\partial X_2} & \cdots & \frac{\partial W}{\partial X_{24}} \\ \frac{\partial W}{\partial X_1} & \frac{\partial W}{\partial X_2} & \cdots & \frac{\partial W}{\partial X_{24}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial W}{\partial X_1} & \frac{\partial W}{\partial X_2} & \cdots & \frac{\partial W}{\partial X_{24}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

where \( W \) is the total strain energy of the frame due to the external loads and redundant forces. The principle is neat and elegant, whereas the numerical calculations involved in the equations above are so cumbersome that it is almost impossible for a structural engineer to reach an exact solution for the system with only a slide rule or desk calculator. To handle a practical problem like this, a grossly simplified model of the actual structure was often used. However, with the advent of the digital computer, the solving of simultaneous equations can now be performed in a matter of seconds.

Example 5-13

Analyze the truss in Fig. 5-18(a). Assume that \( E = 30,000 \text{ kips/in.}^2 \) and \( L \) (ft)/\( A \) (in.\(^2\)) = 1 for all members.

The truss is statically indeterminate to the second degree. We may take bars \( b C \) and \( C d \) as redundant members. As shown in Fig. 5-18(b), these bars are cut and replaced by redundant axial forces \( X_1 \) and \( X_2 \), respectively. The internal force for each bar is then computed in terms of the external load and redundant forces as indicated. The unknowns \( X_1 \) and \( X_2 \) are then solved by the simultaneous equations

\[ \sum \frac{S(\partial S/\partial X_1)L}{AE} = 0 \quad \text{and} \quad \sum \frac{S(\partial S/\partial X_2)L}{AE} = 0 \]

as prepared in Table 5-6. Setting

\[ -78.4 + 4X_1 + 0.64X_2 = 0 \]
\[ 27.2 + 0.64X_1 + 4X_2 = 0 \]

and solving Eqs. 5-38 and 5-39 simultaneously, we obtain

\[ X_1 = +21.2 \text{ kips}, \quad X_2 = -10.2 \text{ kips} \]

The answer for each of the bar forces is given in the last column of Table 5-6. Note that this procedure can be extended to trusses with many redundants.

Structures made up of some members that are two-force members carrying only axial forces and others that are not are called composite structures. They are
conveniently analyzed by the method of least work, as illustrated in the following example.

Example 5-14

Figure 5-19(a) shows a cantilever beam whose other end is supported by a rod. Find the force in the rod. \( E = 30,000 \text{ kips/in}^2 \).

The structure is statically indeterminate to the first degree. Select the force in the tie rod as the redundant \( X \), as shown in Fig. 5-19(b). Then the internal work in the rod is

\[
\frac{X^2 l_1}{2 A_1 E}
\]

and the internal work in the beam is equal to

\[
\int_0^l \left(0.6Xx + \frac{1}{2} \left[0.6Xx - 10(x - 6)\right]^2 \right) dx = \frac{(-0.8X)^3 l_2}{2 A_2 E}
\]

Applying \( \partial W/\partial X = 0 \) gives

\[
\frac{AX_1}{A_1 E} + \int_0^l \left(0.6Xx \right) \left(0.6x \right) dx + \int_0^l \left(0.6Xx - 10(x - 6)\right) \left[0.6x\right] \left[0.6x\right] dx
\]

\[
+ \frac{(-0.8X)(-0.8)l_2}{A_2 E} = 0
\]
applied external force vector \( R \). The external displacements corresponding to \( R \) constitute the nodal displacement vector \( r \), and the external displacement corresponding to \( X \) is the null vector \( 0 \), because the compatibility condition dictates no relative axial movement across the cuts. The external work \( W_E \) is then

\[
W_E = \frac{1}{2} \begin{bmatrix} R \end{bmatrix}^T \begin{bmatrix} r \\ 0 \end{bmatrix}
\]  
(5-41)

Equating the external work to the internal work and canceling the common factor \( \frac{1}{2} \), we obtain

\[
\begin{bmatrix} R \end{bmatrix}^T \begin{bmatrix} r \\ 0 \end{bmatrix} = Q^T q
\]  
(5-42)

Equation 5-42 links four vectors just as Eq. 4-36 of Sec. 4.8b links four vectors.

From equilibrium, we obtain the relationship between the member force \( Q \) and the applied external forces \( R \) and redundants \( X \),

\[
Q = b_R R + b_X X
\]  
(5-43)

where the matrices \( b_R \) and \( b_X \) are the force transformation matrices, which may be obtained by solving the primary structure for each of the applied forces \( R_1, R_2, \ldots \) and \( X_1, X_2, \ldots \), individually either by hand computation or by Gaussian elimination (Appendix A) using the computer. It should be noted that, just as in the method of consistent deformation, the member forces of the cut members must not be forgotten at this step. Equation 5-43 may be cast in the following force transformation form:

\[
Q = [b_R \quad b_X] \begin{bmatrix} R \\ X \end{bmatrix}
\]  
(5-44)

Now, in view of the relationships expressed in Eqs. 5-42 and 5-44, the following displacement transformation equation is directly obtained by contragredient transformation (Sec. 4-8).

\[
\begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} b_R^T q \\ b_X^T \end{bmatrix}
\]  
(5-45)

The first part of Eq. 5-45 gives the nodal displacements \( r \), and the second part is actually the compatibility condition that we are looking for.

Now the elongations are related to the member forces through the total member flexibility matrix \( f \).

\[
q = fQ
\]  
(5-46)

For trusses, the matrix \( f \) is diagonal and contains \( L/EA \) for all members and zero for regular support members or prescribed flexibility constants for elastic supports.
By substituting Eq. 5-46 and then Eq. 5-44 into Eq. 5-45, we obtain

\[
\begin{bmatrix}
  r \\
  0
\end{bmatrix}
= \begin{bmatrix}
  b_r^T \\
  b_x^T
\end{bmatrix} f
\begin{bmatrix}
  R \\
  X
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
  r \\
  0
\end{bmatrix}
= \begin{bmatrix}
  F_{RR} & F_{RX} \\
  F_{XR} & F_{XX}
\end{bmatrix}
\begin{bmatrix}
  R \\
  X
\end{bmatrix}
\tag{5-47}
\]

where

\[
F_{RR} = b_r^T f b_r, \quad F_{RX} = b_r^T f b_x
\]
\[
F_{XR} = b_x^T f b_r, \quad F_{XX} = b_x^T f b_x
\tag{5-48}
\]

The matrix \( F_{RR} \) relates the nodal displacement \( r \) and the nodal force \( R \) of the primary structure and is called the flexibility matrix of the primary structure.

Comparison of the second part of Eq. 5-47 to Eq. 5-9 and Eq. 5-11 reveals that the matrix \( F_{XX} \) is the \( F' \) matrix in Eq. 5-9 and Eq. 5-11, and \( F_{RR} R \) corresponds to \( \Delta' \) in Eq. 5-9 and Eq. 5-11. Obviously, if there are prescribed displacements \( \Delta \) as indicated in Eq. 5-11, the 0 vector in the second part of Eq. 5-47 would be replaced by \( \Delta \). Thus, the second part of Eq. 5-47 is the compatibility condition, which results in the solution of the redundant force vector \( X \):

\[
X = -F_{XX}^{-1} F_{XR} R
\tag{5-49}
\]

The first part of Eq. 5-47 gives the nodal displacement solution:

\[
r = F_{RR} R + F_{RX} X
\tag{5-50}
\]

This equation is also a compatibility statement that relates member deformation to nodal displacement.

Using the expression in Eq. 5-49, we finally reach the force displacement relationship of the indeterminate structure.

\[
r = (F_{RR} - F_{RX} F_{XX}^{-1} F_{XR}) R
\]

or

\[
r = FR
\tag{5-51}
\]

where

\[
F = F_{RR} - F_{RX} F_{XX}^{-1} F_{XR}
\tag{5-52}
\]

is the flexibility matrix of the structure.

This matrix procedure is now demonstrated through an example.

Example 5-15

Find the bar forces of the truss in Fig. 5-20(a) by the force method. Also find the nodal displacement corresponding to the applied load. Assume that \( E = 30,000 \) kips/in.\(^2\) and \( L (\text{ft})/A (\text{in.}^2) = 1 \) for all members.

The truss shown in Fig. 5-20(a) is statically indeterminate to the first degree. Let us select bar \( e \) as the redundant and denote the external load of 12 kips by \( R_e \), as shown in Fig. 5-20(b). The bar forces are denoted by \( Q^e, Q'^e, \ldots, Q'^f \). From equilibrium based on the primary structure of Fig. 5-20(b),

\[
\begin{bmatrix}
  Q^e \\
  Q'^e \\
  Q''^e \\
  Q'^f
\end{bmatrix}
= \begin{bmatrix}
  1 & -\frac{1}{3} & 0 & 0 \\
  -1 & 1 & -\frac{1}{3} & 0 \\
  0 & 0 & -\frac{1}{3} & 0 \\
  0 & 0 & 0 & -\frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
  R_e \\
  X
\end{bmatrix}

b_e \quad b_x
\]

Since \( L/A = 1 \) for all members,

\[
f = \frac{1}{E}
\begin{bmatrix}
  1 \\
  1 \\
  1 \\
  1
\end{bmatrix}
\]

Thus,

\[
F_{XR} = b_x^T f b_e
\]

\[
= \begin{bmatrix}
  -\frac{1}{3} & \frac{1}{2} & -\frac{1}{3} & -\frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
  1 \\
  \frac{1}{E} \\
  0 \\
  0
\end{bmatrix}
= -\frac{3}{3 E}
\]

![Figure 5-20](image)
\[ F_{xx} = b^T f b_x \]
\[ = \begin{bmatrix} -\frac{4}{3} & -\frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \frac{1}{E} \end{bmatrix} = \frac{4}{E} \]

\[ F_{xx}^{-1} = \frac{E}{4} \]

The redundant force \( X \) is then solved by
\[ X = -F_{xx}^{-1} F_{xx} R \]
\[ = -\left(\frac{E}{4}\right) \begin{bmatrix} -\frac{3.3}{E} \end{bmatrix} (12) = 9.9 \text{ kips} \]

Substituting in the equilibrium equation, we obtain
\[ \begin{bmatrix} Q' \end{bmatrix} = \begin{bmatrix} Q' \end{bmatrix} = \begin{bmatrix} 4.08 \\ 3.06 \\ 4.08 \\ -5.94 \\ 9.90 \\ -5.10 \end{bmatrix} \text{ kips} \]

To find \( r_1 \), we first calculate the flexibility matrix of structure \( F \):
\[ F = F_{xx} - F_{xx} F_{xx}^{-1} F_{xx} \]
\[ = \begin{bmatrix} 1 & \frac{4}{3} & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & \frac{1}{E} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{E} \\ 1 & \frac{3.3}{E} \\ 0 & \frac{4}{E} \\ 0 & \frac{3.3}{E} \\ -\frac{1}{E} \end{bmatrix} \]
\[ = \frac{1.4}{E} \]

The displacement \( r_1 \) is then solved:
\[ r_1 = FR_1 \]
\[ = \left(\frac{1.4}{E}\right) (12) = \frac{(1.4)(12)}{30,000} = 0.00056 \text{ ft} \]

in the direction of the applied load.

Note in this example that we did not include the support reactions explicitly as in Sec. 4-8. This is possible because we obtain the solution for \( Q \) with hand calculation, which may not involve the support reactions. Also, we include only the single applied force \( R_1 \) in the formulation of the nodal force vector \( R \), not the complete vector with zero applied forces included, as in Eq. 4-32 of Sec. 4-8. This is allowed because the zero applied forces contribute nothing in the computation. Consequently, however, we are able to solve for \( r_1 \) only. If all nodal displacements are desired, then we need to include even the zero applied nodal forces at the beginning.