Asphalt Pavement Aging and Temperature Dependent Properties through a Functionally Graded Viscoelastic Model – I: Development, Implementation and Verification

Eshan V. Dave, Secretary of M&FGM2006 (Hawaii)
Research Assistant and Ph.D. Candidate

Glaucio H. Paulino, Chairman of M&FGM2006 (Hawaii)
Donald Biggar Willett Professor of Engineering

William G. Buttlar
Professor and Narbey Khachaturian Faculty Scholar

Department of Civil and Environmental Engineering
University of Illinois at Urbana-Champaign
Outline

- Part – I
  - Graded Finite Elements
  - Viscoelasticity and FGMs
  - Finite Element Formulations
  - Verification
  - Concluding Remarks

- Part – II (Companion presentation)
  - Asphalt Pavements
  - Effect of Aging
  - Simulations
  - Concluding Remarks
Objectives

- Develop efficient and accurate simulation scheme for viscoelastic functionally graded materials (VFGMs)
- Correspondence Principle based formulation
- Application: Asphalt concrete pavements (Part II)
Graded Finite Elements

- Graded Elements: Account for material non-homogeneity within elements unlike conventional (homogeneous) elements
    - Direct Gaussian integration (properties sampled at integration points)
  - Kim and Paulino (2002)
    - Generalized isoparametric formulation (GIF)
  - Paulino and Kim (2007) and Paulino et al. (2007) further explored GIF graded elements
    - Proposed patch tests
    - GIF elements should be preferred for multiphysics applications
  - Buttlar et al. (2006) demonstrated need of graded FE for asphalt pavements (elastic analysis)
Generalized Isoparametric Formulation (GIF)

- Material properties are sampled at the element nodes
- Iso-parametric mapping provides material properties at integration points
- Natural extension of the conventional isoparametric formulation

Material Properties (eg. $E(x, y)$)

$$E(x, y) = E_0 \exp[3x - 2y]$$

$$E = \sum_{i=1}^{m} N_i E_i$$

$N_i = \text{Shape function corresponding to node, } i$

$m = \text{Number of nodes per element}$

$\mathbf{z} = E(x, y)$
Viscoelasticity: Basics

- Constitutive Relationship for linear viscoelastic body:

\[
\sigma_{ij}^d(x,t) = 2 \int_{t'=-\infty}^{t=t} G_{ijkl}(x,\xi(t) - \xi(t')) \varepsilon_{kl}^d(x,t') \, dt'
\]

\[
\sigma_{kk}(x,t) = 3 \int_{t'=-\infty}^{t=t} K_{kkll}(x,\xi(t) - \xi(t')) \varepsilon_{ll}(x,t') \, dt'
\]

- \(\sigma_{ij}\) are stresses, \(\varepsilon_{ij}\) are strains
- Superscript \(d\) represents deviatoric components
- \(G_{ijkl}\) and \(K_{ijkl}\): shear and bulk moduli (space and time dependent)
- Assumptions: no body forces, small deformations
- Equilibrium: \(\sigma_{ij,j} = 0\)
- Strain-Displacement: \(\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\)
- \(u_i\): displacements
Viscoelasticity: Correspondence Principle

- Correspondence Principle (Elastic-Viscoelastic Analogy): “Equivalency between transformed (Laplace, Fourier etc.) viscoelastic and elasticity equations”

**Elastic**
\[
\sigma^d = 2G\varepsilon^d \\
\sigma = 3K\varepsilon
\]

**Transformed Viscoelastic**
\[
\tilde{\sigma}^d = 2\tilde{G}\tilde{\varepsilon}^d \\
\tilde{\sigma} = 3\tilde{K}\tilde{\varepsilon}
\]

- Extensively utilized to solve variety of nonhomogeneous viscoelastic problems:
  - Hilton and Piechocki (1962): *Shear center of non-homogeneous viscoelastic beams*
  - Chang et al. (2007): *Thermal stresses in graded viscoelastic films*
Viscoelastic Model

- Prony series form: Generalized Maxwell Model
  - Equivalency between compliance and relaxation forms
  - Flexibility in fitting experimental data
  - Transformations are well established
  - Readily applicable to asphaltic and other viscoelastic materials (polymers, etc)

\[
E(t) = \sum_{i=1}^{h} E_i \exp\left[-t / \tau_i\right]
\]

\[
\tau_i = \frac{\eta_i}{E_i}
\]
Viscoelastic FGMs

- Paulino and Jin (2001); Mukherjee and Paulino (2003)
  - Material with “Separable Form”

\[ E(x, t) = E(x)f(t) \]
General FE Implementation

- Correspondence principle based implementation using Laplace transform (Yi and Hilton, 1998)
- Variational Principle (Potential): (Taylor et al., 1970)

\[
\Pi = \int_{\Omega} \int_{t''=-\infty}^{t''=t} \int_{t'=-\infty}^{t'=t} \frac{1}{2} C_{ijkl} \left[ x, \xi_{ijkl}(t - t'') - \xi'_{ijkl}(t') \right] \frac{\partial \varepsilon_{ij}(x, t')}{\partial t'} \frac{\partial \varepsilon_{kl}(x, t'')}{\partial t''} \, dt' \, dt'' \, d\Omega
\]

\[
- \int_{\Omega} \int_{t''=-\infty}^{t''=t} \int_{t'=-\infty}^{t'=t} C_{ijkl} \left[ x, \xi_{ijkl}(t - t'') - \xi'_{ijkl}(t') \right] \frac{\partial \varepsilon^*_{ij}(x, t')}{\partial t'} \frac{\partial \varepsilon^*_{kl}(x, t'')}{\partial t''} \, dt' \, dt'' \, d\Omega
\]

\[
- \int_{S} \int_{t''=-\infty}^{t''=t} P_i(x, t - t'') \frac{\partial u_i(x, t'')}{\partial t''} \, dt'' \, dS
\]
**FE Implementation: Basis**

- **Stationarity:**

\[
\delta \Pi = \int \int \int_{\Omega_u} \left\{ C_{ijkl} \left[ x, \xi_{ijkl} (t-t^*) - \xi'_{ijkl} (t^*) \right] \frac{\partial}{\partial t} \left( \varepsilon_{ij} (x, t^*) - \varepsilon^*_{ij} (x, t^*) \right) \frac{\partial \delta \varepsilon_{kl} (x, t^*)}{\partial t^*} \right\} dt \, dt^* \, d\Omega_u
\]

\[- \int \int \int_{\Omega_\sigma} P_i (x, t-t^*) \frac{\partial \delta u_i (x, t^*)}{\partial t^*} dt^* \, d\Omega_\sigma = 0.\]

\[\Omega: \text{ volume, } S \text{ surface with traction } P_i\]

\[C_{ijkl}: \text{ constitutive properties}\]

\[\varepsilon_{ij}: \text{ mechanical strains, } \varepsilon^*_{ij}: \text{ thermal strains, } u_i: \text{ displacements,}\]

\[\xi: \text{ reduced time related to real time through time-temperature superposition principle given by:}\]

\[\xi(t) = \int_0^t a(T(t')) dt'\]

\[a \text{ is time-temperature shift factor, and } T \text{ is temperature}\]
**FEM**

- **Element stiffness matrix:**

  \[ k_{ij}(x,t) = \int_{\Omega_u} B_{ik}^T(x)C_{kl}(x,\xi(t))B_{lj}(x)\,d\Omega_u \]

- **Force vectors:**

  **Mechanical:**

  \[ f_i(x,t) = \int_{\Omega_\sigma} N_{ij}(x)P_j(x,t)\,d\Omega_\sigma \]

  **Thermal:**

  \[ f_i^{th}(x,t) = \int_{\Omega_u} \int_{-\infty}^{t} B_{ik}(x)C_{kl}(x,\xi(t)-\xi(t'))\frac{\partial\varepsilon_i^*(x,t')}{\partial t'}\,dt'\,d\Omega_u \]

  \[ k_{ij} : \text{element stiffness matrix,} \quad N_{ij} \quad \text{shape functions} \quad B_{ij} \quad \text{derivatives} \]

  \[ f_i : \text{element force (load) vector} \quad u_i \quad \text{displacement vector} \]

  \[ \varepsilon_i : \text{strains related to nodal degrees of} \quad q_j \quad \text{through isoparametric} \]

  \[ u_i(x,t) = N_{ij}(x)q_j(t) \]

  \[ \varepsilon_i(x,t) = B_{ij}(x)q_j(t) \]
Assembling provides global stiffness matrix, $K_{ij}$ and force vectors, $F_i$

Equilibrium:

$$K_{ij}(x, \xi(t))U_j(0) + \int_{0^+}^{t} K_{ij}(x, \xi(t) - \xi(t')) \frac{\partial U_j(t')}{\partial t'} dt' = F_i(x,t) + F_i^{th}(x,t)$$

Correspondence principle:

$$\tilde{K}_{ij}(x,s)\tilde{U}_j(s) = \tilde{F}_i(x,s) + \tilde{F}_i^{th}(x,s)$$

$a(s)$ is Laplace transform of $a(t)$, $s$ is transformation variable

$$\tilde{a}(s) = \int_{0}^{\infty} a(t) \exp(-st) dt$$
FEM: Implementation

Define problem in time-domain (evaluate load vector, $F(x, t)$ and stiffness matrix components $K(x)$ and $\Lambda(t)$)

Perform Laplace transform to evaluate $\tilde{F}(x, s)$ and $\tilde{\Lambda}(s)$

Solve linear system of equations to evaluate nodal displacement, $\tilde{U}(x, s)$

Perform inverse Laplace transforms to get the solution, $U(x, t)$

Post-process to evaluate field quantities of interest
FEM: Verification

- MATLAB® code using GIF and correspondence principle
- GIF
  - Compare analytical and numerical solutions for graded boundary value problems
- Viscoelasticity
  - Compare analytical and numerical solutions for viscoelastic bar imposed with creep loading
- Comparison with Commercial Code ABAQUS® (Layered Approach)
Graded Finite Element Performance

Bending example

Analytical Solution (line)

Numerical Solution (markers)

Stress, $\sigma_{yy}$ (MPa)

$x$ (mm)

$E(x) = E_0 \exp[\beta x]$

$E_0 = 100$

$\beta = 2$

$\sigma_b = 10$
Homogeneous Viscoelastic Verification

Creep example shown here

Numerical inversion performed with 20-Collocation points

\[ \sigma(t) = E_0 \exp\left(-\frac{t}{\tau}\right) \]

\( E_0 = 10 \)
\( \tau = 20 \)
FGM Verification with ABAQUS

- Simply supported beam in 3-point bending
  - 100-second creep loading
- Graded viscoelastic material properties

FE simulation:
- Homogeneous: Averaged properties
- Layered (ABAQUS):
  - 6-Layers
  - 12-Layers
- Graded:
  - Same mesh structure as 6-Layers
Reference Material Properties
FEM Meshes

- 6-Layers / FGM / Homogeneous
  - 3146 DOFs
  - 6-node triangle elements

- 12-Layers
  - 6878 DOFs
  - 6-node triangle elements
Concluding Remarks

- Main Contribution: development of graded viscoelastic elements
- Extension of the Generalized Isoparametric Formulation (Elastic) to rate-dependent materials (viscoelastic)
- Correspondence Principle based formulation: separable material properties
- Companion presentation (paper) demonstrates application of this work to field of asphalt pavements
- Extension: Graded Viscoelastic formulation in time domain
Thank you for your attention!!