Finite Element Method of Multiscale Type
Basic Ideas and Applications

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Multiple Scale Phenomena

• PDEs with highly oscillatory coefficients \iff Problems in heterogeneous materials

• Different scales \textit{in the domain itself}, like plates and shells, or domains with rough boundary

• Reaction dominated reaction–diffusion eqtns
Decomposition

In general we decompose the solution as:

\[ u_{\text{solution}} = u_{\text{macro}} + u_{\text{micro}} \]

- Aim of multiscale modeling: macroscopic behaviour without resolving the microscale features.
- Multiscale Finite Element Method: decompose

\[ u_{\text{MsFEM}} = u_{\text{linear}} + u_{\text{Ms}} \]

where \( u_{\text{linear}} \) is piecewise linear, and \( u_{\text{Ms}} \) brings information about the microscales.
Consider the problem

\[ \mathcal{L}^\varepsilon u = f \quad \text{in } \Omega, \]

\[ u = 0 \quad \text{on } \partial\Omega, \]

and its weak formulation: find \( u \in H^1_0(\Omega) \) such that

\[ a(u, v) = (f, v) \quad \text{for all } v \in H^1_0(\Omega). \]

Here, \( \Omega \) is a polygon, \( \varepsilon > 0 \) is a small parameter, and

\[ (f, v) = \int_\Omega fv \, dx \]
Example I: Thermal problem

\[ \mathcal{L}^\varepsilon u := - \text{div} (K(x, \varepsilon) \text{grad } u), \]

and

\[ a(u, v) = \int_{\Omega} (K(x, \varepsilon) \text{grad } u) \cdot \text{grad } v \, dx. \]

Example II: Reaction–diffusion problem

\[ \mathcal{L}^\varepsilon u := -\varepsilon \Delta u + u, \]

and

\[ a(u, v) = \int_{\Omega} \varepsilon \text{grad } u \cdot \text{grad } v + uv \, dx. \]
Residual Free Bubbles (RFB)

Consider a partition of the domain $\Omega$ into finite elements, and the associated \textit{enriched space}

$$V_h := V_1 \oplus B,$$

where

- $V_1 \subset H_0^1(\Omega)$ is the space of piecewise linear or bilinear functions
- $B \subset H_0^1(\Omega)$ is the space of “bubbles”, functions that vanish over the edges of the finite elements
The method consists in finding \( u_h \in V_h = V_1 \oplus B \) where

\[
a(u_h, v) = (f, v) \quad \text{for all } v \in V_h.
\]

Writing \( u_h = u_1 + u_b \) implies

\[
a(u_1 + u_b, v_1) = (f, v_1) \quad \text{for all } v_1 \in V_1,
\]
\[
a(u_1 + u_b, v_b) = (f, v_b) \quad \text{for all } v_b \in B.
\]

Hence, the second equation holds elementwise:

\[
a(u_1 + u_b, v_b)|_K = (f, v_b)|_K \quad \text{for all } v_b \in H^1_0(K),
\]

for every element \( K \).
The bubble is the strong solution of the local problem

$$\mathcal{L}^\varepsilon u_b = -\mathcal{L}^\varepsilon u_1 + f \quad \text{in } K,$$

$$u_b = 0 \quad \text{on } \partial K.$$

Write $$u_b = T(-\mathcal{L}^\varepsilon u_1 + f)$$ and do static condensation:

$$a(u_1 + u_b, v_1) = (f, v_1)$$

$$\implies a(u_1 + T(-\mathcal{L}^\varepsilon u_1 + f), v_1) = (f, v_1)$$

$$\implies a(u_1 - T\mathcal{L}^\varepsilon u_1, v_1) = (f, v_1) - a(Tf, v_1)$$

$$\implies a((I - T\mathcal{L}^\varepsilon)u_1, v_1) = (f, v_1) - a(Tf, v_1)$$

for all $$v_1 \in V_1,$$
First point of view

We can see this formulation as a Parameter Free Stabilized Method:

Search for \( u_1 \in V_1 \) where

\[
a(u_1, v_1) - a(T \mathcal{L}^\varepsilon u_1, v_1) = (f, v_1) - a(Tf, v_1)
\]

for all \( v_1 \in V_1 \).
Second point of view

We can see this formulation as a *numerical upscaling procedure*:

Search for $u_1 \in V_1$ where

$$a^*(u_1, v_1) = \langle f^*, v_1 \rangle$$

for all $v_1 \in V_1$,

and

$$a^*(u_1, v_1) = a((I - T \mathcal{L}^\varepsilon)u_1, v_1), \quad \langle f^*, v_1 \rangle = (f, v_1) - a(Tf, v_1).$$

Multiscale interpretation:

- $V_1$ is the coarse space, seeing only the “macro” properties
- $V_B$ is the fine space, capturing the small scale features
Third point of view

We can see this formulation almost like a Petrov–Galerkin Method:

If \( \{ \psi_i \} \) is a basis of \( V_1 \), and \( u_1 = \sum_{i=1}^{N} u_i \psi_i \), then

\[
\sum_{i=1}^{N} u_i a((I - T \mathcal{L}^\varepsilon)\psi_i, \psi_j) = (f, \psi_j) - a(Tf, \psi_j)
\]

\[
\Rightarrow \sum_{i=1}^{N} u_i a(\lambda_i, \psi_j) = (f, \psi_j) - a(Tf, \psi_j), \text{ where } \lambda_i = (I - T \mathcal{L}^\varepsilon)\psi_i.
\]

Hence,

\[
\mathcal{L}^\varepsilon \lambda_i = 0 \quad \text{in } K, \quad \lambda_i = \psi_i \quad \text{on } \partial K,
\]

The basis functions of the trial space solve the operator locally, and the test functions remain the same.
Upset:

The RFB strategy works pretty well for second order PDEs with oscillatory coefficients (Sangalli, 2003), but fails for the reaction–diffusion equation.
Example: Consider the domain $\Omega = (0, 1) \times (0, 1)$ and the problem

$$-10^{-6} \Delta u + u = 1 \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega,$$
The standard piecewise linear Galerkin approximation is given by
The RFB approximation is given by

and the spurious oscillations are still there.
So, what goes wrong?

- Imposing that the bubbles vanish at each element edge causes the functions in the enriched space to be linear over the edges, and hence they are unable to capture the boundary layer effects.

- We use then an idea by Tom Hou and X.H. Wu (JCP, 1997), and impose that the basis functions solve the operator inside each element, and solve an ODE over each edge. This ODE is defined using a “1D restriction” of the original operator.
New idea: enriching the finite element space with local, but not bubble-like function

We want $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h$$

1. Enrich the trial space $U_h$ with local solutions with boundary values determined by edge restrictions of the governing differential operator

2. Enrich the test space $V_h$ with residual-free bubble functions

Therefore we start out with a Petrov-Galerkin setting.
After some formalism (similar to RFB), we gather that the nodal values \( u_i \) solve

\[
\sum_{i=1}^{N} u_i a(\theta_i, \psi_j) = (f, \psi_j) - a(Tf, \psi_j),
\]

where \( \mathcal{L}^\varepsilon \theta_i = 0 \) in each element. To determine \( \theta_i \) use the boundary condition

\[-\varepsilon \partial_{ss} \theta_i + \sigma \theta_i = 0 \quad \text{over the edges}\]

\[\theta_i = 1 \text{ at the } i\text{th node}, \quad \theta_i = 0 \text{ at the other nodes},\]

and \( s \) is the variable running along the edge.

We do have analytic expressions for \( \theta_i \).
In the simplest case (a square):

\[
\theta(x, y) = \frac{\sinh \left( \sqrt{\frac{1}{2\varepsilon}} h \left(1 - \frac{x}{h}\right) \right) \sinh \left( \sqrt{\frac{1}{2\varepsilon}} h \left(1 - \frac{y}{h}\right) \right)}{\sinh \left( \sqrt{\frac{1}{2\varepsilon}} h \right) \sinh \left( \sqrt{\frac{1}{2\varepsilon}} h \right)}
\]
Typical basis functions $\theta$ for $\varepsilon = 1.0$:
Typical basis functions $\theta$ for $\varepsilon = 0.1$: 
Typical basis functions $\theta$ for $\varepsilon = 10^{-3}$:
Numerical Results
Example I: Consider $\Omega = (0, 1) \times (0, 1)$, $f = 1$ with $u = 0$ on $\partial \Omega$, and $\varepsilon = 10^{-6}$:
Linear Galerkin:
RFB:
New Formulation
**NACA Example:** Let $f = 0$, and $u = 0$ on the outer boundary and $u = 1$ in the inner boundary ($\varepsilon = 10^{-6}$):
Isovalues of the solutions by Galerkin method:
Isovalues of the solutions by the enriched method:
The new method captures the boundary layer accurately. Zoom of isovalues:
Solution profile:

- GALERKIN METHOD
- UNUSUAL METHOD
- NEW ENRICHED METHOD
Conclusions

• Inovative finite element methods have the potential to solve accurately problems where multiple scales play a significant role.

• The Residual Free Bubbles approach fails for reaction-diffusion eqtns. The culprit is the restriction that bubbles should vanish on element edges.

• We propose a new Petrov-Galerkin formulation eliminates the zero edge condition.